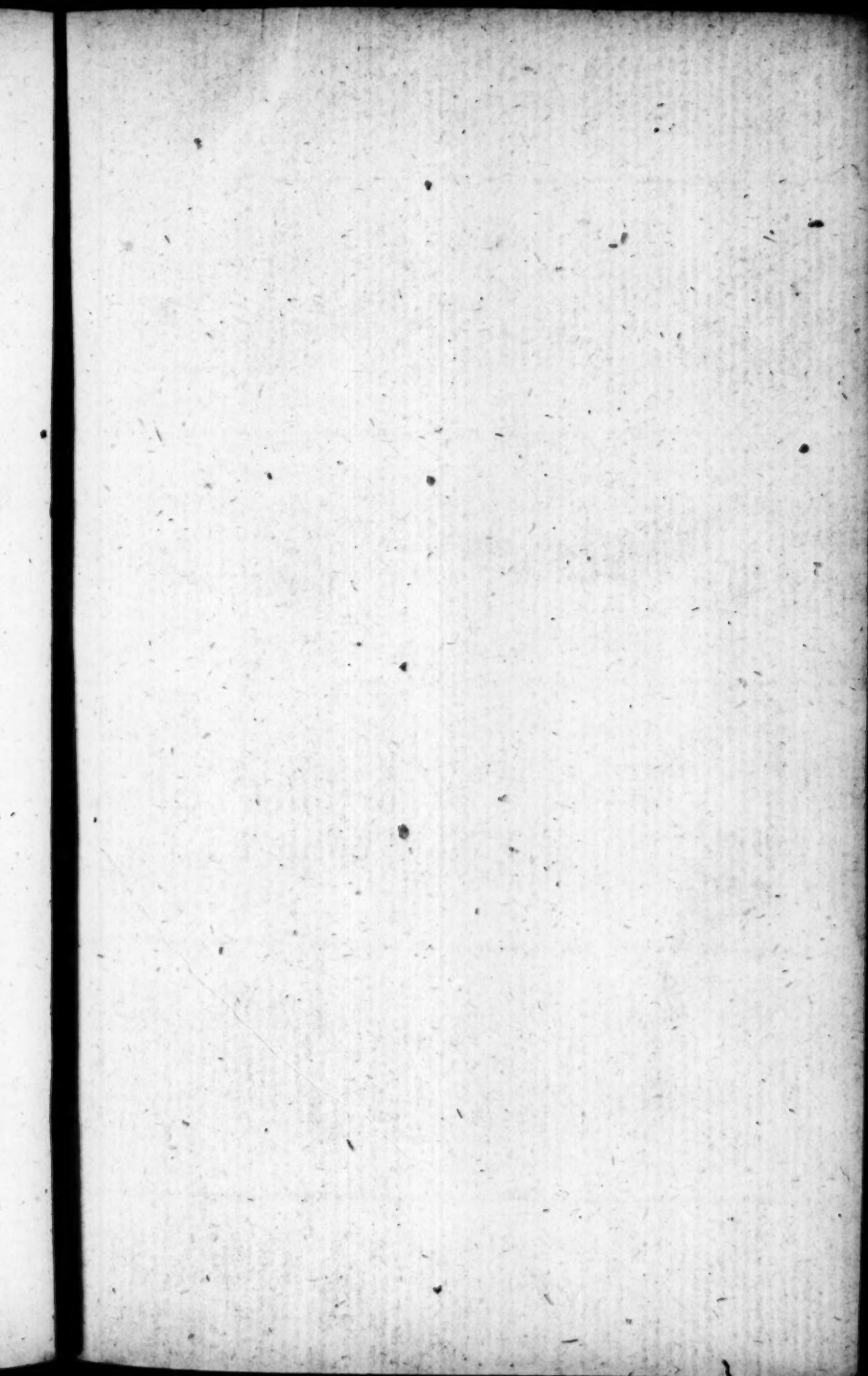
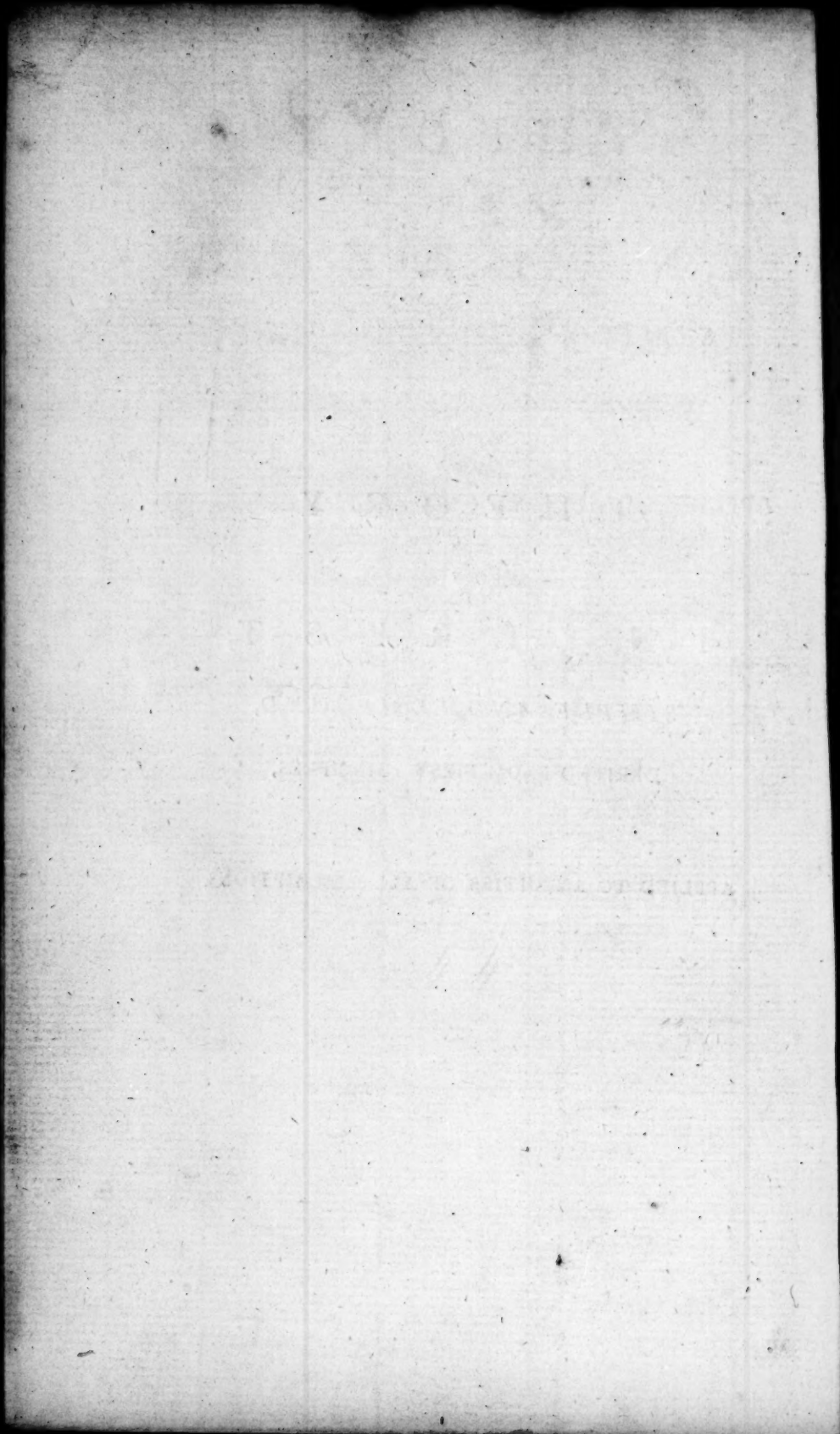




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T H E O R Y
O F
I N T E R E S T,
SIMPLE AND COMPOUND,
DERIVED FROM FIRST PRINCIPLES,
A N D
APPLIED TO ANNUITIES OF ALL DESCRIPTIONS.



THEORY OF INTEREST, SIMPLE AND COMPOUND,

DERIVED FROM FIRST PRINCIPLES,

AND
APPLIED TO ANNUITIES OF ALL DESCRIPTIONS.

CONTAINING,

Arithmetical and Geometrical Progressions.	Annuities on Lives, with Tables.
Simple and Compound Interest, with Tables.	M. De Moivre's Hypothesis of Equal Decrements, with 22 Problems, shewing the Value of Life in all its Varieties.
Annuities at Simple and Compound Interest, with Tables.	Insurance on Lives.
Of a Sinking Fund to extinguish the National Debt.	Of the Expectation of Life.
The Value of a Perpetuity.	Of the British State Lottery.
	Of Tontines, and Insurance from Fire; &c.

WITH
AN ILLUSTRATION OF THE WIDOWS SCHEME
IN THE CHURCH OF SCOTLAND.

By THE REVEREND MR DAVID WILKIE,
MINISTER OF CULTS.

EDINBURGH:

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AND G. G. & J. ROBINSON, *London*.

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DEDICATION.

TO THE RIGHT HONOURABLE,

FRANCIS Lord NAPIER.

MY LORD,

IT may be unusual, that a person who hath not had the honour of your Lordship's acquaintance, should, in this public manner, crave your protection, in his attempt to promote the knowledge of that science, whereby human life, which is liable to so many accidents, is yet, in some measure, reduced to rule. It flows, my Lord, from the respect which I entertain for the memory of your illustrious progenitor, John Napier, Baron of Merchiston, in Scotland, whom I place
upon

upon an equal footing with the most distinguished characters of the seventeenth century, for his valuable invention of Logarithms, that I presume to ask your patronage in the publication of the following Treatise, and to profess myself, with all respect,

My LORD,

Your Lordship's most obedient,

And humble servant,

DAVID WILKIE.

PRE-

THE PREFACE.

SIMILAR to the satisfaction which the mind receives in the investigation of geometrical propositions, is the pleasure which we derive from the resolution of questions in algebra, both on account of their ingenuity, certainty, and extensive usefulness. In the following Treatise, wherein so many algebraic theorems occur, it is proposed to investigate, from first principles, the rules adapted to one particular branch of science, that of Interest, both simple and compound, with their application to Annuities of all descriptions; hoping thereby, not only to afford pleasure to the studious, in the investigation of the theorems, but advantage also to men of business, in the application of these to the affairs of property.

The theorems, such as $\frac{a}{r}$ being the value of a perpetuity, are as beautiful as any which are to be met with in the other branches of science; and to render them the more intelligible, a demonstration is subjoined where it is thought necessary. The tables are accurate, and several of them new, such as Table IV. VIII. and XI. of annuities on lives; the first expressing the amount of L. 1 annuity upon a single life; in the construction

tion of which, the reason is discovered, why the value of a single life is less, and its amount greater, than that corresponding to its expectation; and the two last shewing the probabilities of life adapted to Scotland. The construction of the tables is clearly pointed out; and, by three equations annexed to each, all their uses are discovered at first sight. Several branches of education, connected with this science, are treated of, such as Arithmetical and Geometrical Progressions, the Extraction of the Square Root, Logarithms, the Arithmetic of Infinities, Fluxions, and the rule of Double Position.

In this Treatise there are many things new. The application of geometry, for amusement, to matters of interest, most of the remarks annexed to each subject, the bearer's loss in discounting bills, several of the theorems, such as $P = \frac{PR - np}{nr}$, expressing the value of a single life, a Tontine, and Insurance from Fire, are all new. But what I chiefly value in this respect, are Prob. III. and XVIII. of Annuities on Lives; the former discovering the value, and the latter the expectation of the joint continuance, or of the longest liver of any number of lives, by methods which are both accurate, and save many pages of algebraic investigation. As the questions in Annuities on Lives are various as the complections of men, I have,

in

n the course of twenty-two problems, delineated a great number of those which are marked with the strongest features. Particular attention hath been paid to the Widows Scheme in the Church of Scotland, wherein so many are interested; and the account given of its construction and operation may enable Contributors to judge for themselves, should the affairs of the Fund ever come before them in a judicial capacity.

In the construction of the theorems of Interest and Annuities certain, I acknowledge myself much indebted to the labours of that able algebraist Mr Nicolas Vilant, St Andrew's. Wherein I differ from others, as in the construction of a widows scheme, in Prob. VIII. let the learned judge for themselves; and wherein I am defective, I crave their indulgence. But I hope it will not be reckoned a defect, that I have not expressed the theorems in so many words, being persuaded that an algebraic expression is much more intelligible at first sight, than it can be made by any language.

The science of Annuities on Lives is yet in its infancy. Dr Halley was the first who constructed a table of the Probabilities of Life: Moivre, Dodson, Webster, and Price, these fathers of the science, dropped only a few years ago. Those theo-

remains of Mr De Moivre, which are built upon the principle, "that the decrements of life are in a constant ratio," though beautiful in theory and easy in practice, are now exploded, because they give the value of life too small. In Scotland, we have no writer who treats methodically of Annuities; and English authors on these subjects are in very few hands: therefore, till a better appears, it is hoped that this Treatise, which has for its object a more perfect System of Interest and Annuities than has hitherto appeared, will not be deemed an unnecessary performance.

CULTS, }
October 18. 1792. }

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The

The Signs which occur, without being explained in the body of the Work, are such as these :

= The Sign of Equality.

+ ——— Addition.

— ——— Subtraction.

× ——— Multiplication.

$\frac{a}{b}$, or $a \div b$ — Division; a divided by b .

$\overline{a+b}$, The Sum of the two Quantities a and b .

$a \oslash b$, The Difference of the two Quantities a and b .

$a : b :: c : d$, Ratio; a is to b as c is to d ,

$\sqrt{a}$, The Square Root of a .

$\sqrt[n]{a}$, The n Root of a .

$\overline{a+b}^2$, The Square of $\overline{a+b}$.

R^n , The n Power of R .

$\frac{1}{R^n}$, 1 divided by the n power of R .

L. The Logarithm of a Quantity.

CHAPTER I.

Of ARITHMETICAL PROGRESSION.

1. **W**HEN a series of quantities, exceeding two in number, increases or decreases by the addition or subtraction of one common difference, the terms of the series are said to be in continued arithmetical progression. Thus,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10. an increasing series.
 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. a decreasing series.

$$a, a+d, a+2d, a+3d, a+4d \dots a+n-1 \times d = v.$$

$$v, v-d, v-2d, v-3d, v-4d \dots v-n-1 \times d = a.$$

$$a+v, a+v, a+v, a+v, a+v \dots \quad a+v = \left\{ \begin{array}{l} \text{Sum of} \\ \text{both.} \end{array} \right.$$

2. If four quantities are thus proportional, the sum of the extremes will be equal to the sum of the means; Thus,

$$a, a+d, a+2d, a+3d: \text{ then } 2a+3d = \left\{ \begin{array}{l} \text{Sum of the extremes.} \\ \text{Sum of the means.} \end{array} \right.$$

3. If three quantities are thus proportional, the
A sum

sum of the extremes is double of the mean;
Thus,

$$a, a+d, a+2d; \text{ then } 2 a+d = \begin{cases} \text{Sum of the extremes.} \\ \text{Double of the mean.} \end{cases}$$

4. Whatever be the number of terms in an arithmetical progression, the sum of the two extremes will be equal to the sum of any two terms equidistant from these extremes. Thus, $1+10 = 5+6 = 3+8$.

5. In a series of terms in arithmetical progression, let

a = the first term.

v = the last term.

n = the number of terms.

d = the common difference.

s = the sum of the series; then as d is found in all the terms except the first, its co-efficient in the last term will be $n-1$, hence $a + \overline{n-1} \times d = v$, $v - \overline{n-1} \times d = a$, and $\overline{n-1} \times d = a \cap v$.

6. In any number of terms in arithmetical progression, the first and last terms, multiplied by the number of terms, are equal to double the sum of the whole series; Thus in the two series above (1) which are perfectly equivalent, it is manifest that the two taken together, that is the double of any one of them, will be equal to the sum of a series of equals $\overline{a+v}$, $\overline{a+v}$ &c. taken to n terms $= \overline{a+v} \times n = 2 s$.

7. Therefore, in a continued arithmetical progression, if any three of the five quantities, a , v , d , n , and s , are supposed to be given, the remaining two may be found, as in the following table:

Given.

Given.		
$a, d, n,$	$s = n \times \overline{a + d \times \frac{n-1}{2}}, \dots$	Equation 1.
$a, d, v,$	$s = \frac{v+a}{2} \times \frac{v-a}{2} + 1, \dots$	2.
$a, n, v,$	$s = \overline{a + v \times \frac{n}{2}}, \dots$	3.
$d, n, v,$	$s = n \times \overline{v - d \times \frac{n-1}{2}}, \dots$	4.
$a, d, n,$	$v = \overline{a + n-1 \times d}, \dots$	5.
$a, d, s,$	$v = \sqrt{a \times \overline{a-d} + d \times 2s + \frac{d}{4}} - \frac{1}{2}d, \dots$	6.
$a, n, s,$	$v = \frac{2s}{n} - a, \dots$	7.
$d, n, s,$	$v = \frac{s}{n} + \frac{1}{2}d \times \overline{n-1}, \dots$	8.
$d, v, n,$	$a = \overline{v - n-1 \times d}, \dots$	9.
$d, n, s,$	$a = \frac{s}{n} - \frac{1}{2}d \times \overline{n-1}, \dots$	10.
$n, v, s,$	$a = \frac{2s}{n} - v, \dots$	11.
$d, v, s,$	$a = \frac{1}{2}d \pm \sqrt{v \times \overline{v+d} - d \times 2s + \frac{d}{4}}, \dots$	12.

in Equation 1st.

$$s = n \times \overline{a + v}$$

$$s = n \times \overline{\frac{a+v}{2}}$$

$$v = \overline{a + d \times n-1}$$

$$s = n \times \overline{\frac{2a + d \times n-1}{2}}$$

$$= n \times \overline{a + d \times \frac{n-1}{2}}$$

Sought v in Equation 6.

$$\overline{a + v \times n} = 2s$$

Sec. 5. $n = \frac{v-a}{d} + 1$

$$\frac{v^2 - a^2}{d} + v + a = 2s$$

$$v^2 - a^2 + dv + da = 2ds$$

$$v^2 + dv = a \times \overline{a-d} + 2ds$$

$$v^2 + dv + \frac{1}{4}d^2 = a \times \overline{a-d} + d \times 2s + \frac{d}{4}$$

$$v + \frac{1}{2}d = \sqrt{a \times \overline{a-d} + d \times 2s + \frac{d}{4}}$$

A 2

Given.

Given.	
$a, d, v,$	$n = \frac{v-a}{d} + 1, \dots$ Equation 13.
$a, d, s,$	$n = \sqrt{\frac{2s-a}{d} + \frac{a^2}{d^2} + \frac{1}{4}} - \frac{a}{d} + \frac{1}{2}, \dots$ 14.
$a, v, s,$	$n = \frac{2s}{v+a}, \dots$ 15.
$d, v, s,$	$n = \frac{v}{d} + \sqrt{\frac{v^2}{d^2} + \frac{v}{d} + \frac{1}{4} - \frac{2s}{d} + \frac{1}{2}}, \dots$ 16.
$a, v, s,$	$d = \frac{v+a \times v-a}{2s-v+a}, \dots$ 17.
$a, n, s,$	$d = \frac{1}{n-1} \times \frac{2s-2an}{n}, \dots$ 18.
$a, n, v,$	$d = \frac{v-a}{n-1}, \dots$ 19.
$n, v, s,$	$d = \frac{2v-2s\frac{1}{n}}{n-1}, \dots$ 20.

Sought n in Equation 14th.	
Sec. 5. and 6.	$2a + dn - d \times n = 2s$ $2an + dn^2 - dn = 2s$ $n^2 + \frac{2a}{d}n - n = \frac{2s}{d}$ $n + \frac{a}{d} - \frac{1}{2} = \sqrt{\frac{2s-a}{d} + \frac{a^2}{d^2} + \frac{1}{4}}$
Sought n in 16th.	
Sec. 5. and 6.	$2vn - dn - dn^2 = 2s$ $\frac{2v}{d}n + n - n^2 = \frac{2s}{d}$ $\left. \begin{aligned} &\frac{v}{d} + \frac{1}{2} - \frac{2v}{d} \\ &\frac{v}{d} + \frac{1}{2} - n \end{aligned} \right\} n + n^2 = \frac{v}{d} + \frac{1}{2} - \frac{2s}{d}$ $\frac{v}{d} + \frac{1}{2} - n = \&c.$
Sought d in 17th.	
Sec. 6.	$\frac{v+a \times v-a}{d} + 1 = 2s$ $\frac{v+a \times v-a}{d} + v + a = 2s$ $v+a \times v-a = 2ds - d \times v + a$
Sought d in 18th.	
	$2a + \frac{1}{n-1} \times d \times n = 2s$ $2an + \frac{1}{n-1} \times dn = 2s$ $\frac{1}{n-1} \times d = \frac{2s-2an}{n}$
Sought d in 20th.	
	$2v - n - 1 \times d \times n = 2s$ $2vn - n - 1 \times dn = 2s$ $\frac{1}{n-1} \times d = 2v - 2s\frac{1}{n}$

In the above table, the expressions belong to an increasing progression, and may be applied also to a decreasing one, by making the quantities a and v change places every where.

EXAMPLES.

1st, Sought, the sum of the first hundred numbers in their natural order. Here, $a=1$, $n=100$, $v=100$; by equation 3^d, $\frac{a+v}{2} \times n = 101 \times 50 = 5050 = \text{sum sought.}$

2^d, Suppose a basket and 500 apples were placed in a straight line, a yard distant from each other, it is required how far one must go before he brings them, one by one, into the basket? Here, as the distance must be gone over twice in bringing each apple, $a=2$, $d=2$, $n=500$; by equation 1st, $n \times \frac{a+d}{2} \times \frac{n-1}{2} = 500 \times 501 = 250500 \text{ yards} = 142 \text{ miles.}$

3^d, A man had 12 children; the youngest was three years old, and the common difference of their ages was four years; What was the age of the eldest? Here $a=3$, $d=4$, $n=12$. By equation 5th, $a + \frac{n-1}{2} \times d = 3 + 44 = 47 \text{ years.}$

8. When a number of terms in arithmetical progression begins with a cypher, (0), which is the most regular series, in this case, the sum of the series is equal to the last term multiplied by half the number of terms. For $a=0$, and, equation

tion $3d, s = v \times \frac{1}{2}$. Thus 0, 1, 2, 3, 31
 $= 31 \times 16 = 496$.

9. When $a=2$, and $d=2$ also, in this case, in equation 1st, $s = n^2 + n = a$ pronic number, which is produced by the addition of even numbers in an arithmetical progression, beginning at 2; and the pronic root $n = \frac{\sqrt{4s+1}-1}{2}$.

10. When $a=1$ and $d=2$, then $s=n^2$. So that all square numbers may be considered as arising from the addition of odd numbers in an arithmetical progression. Thus.

Arithmetical Progression, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, &c.

Square Numbers, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, &c.

Hence may be determined how many odd numbers must be added to produce a given power of that number. Let the given number be n , the exponent of its power m , and the first term of the series a ; then since $d=2$, and the number of terms n , hence $s = n^m = n \times a + n-1$ and $a = n^{m-1} - n + 1$. And thus it appears, how powers of all dimensions may be obtained by summing up of odd numbers. Thus, supposing $m=3$, and $n=3, 4, 5, 6$, &c.

$$3^3 = 7 + 9 + 11$$

$$4^3 = 13 + 15 + 17 + 19.$$

$$5^3 = 21 + 23 + 25 + 27 + 29.$$

$$6^3 = 31 + 33 + 35 + 37 + 39 + 41.$$

CHAP.

AND ANNUITIES.

C H A P. II.

Of SIMPLE INTEREST.

11. **W**HEN a sum of money is lent upon interest, and that interest is either kept in the borrower's hands, or is paid regularly, without becoming a part of the principal, it is then said to bear simple interest.

The interest of money was formerly very high; but by an act of parliament, made in 12th Queen Anne, it is now reduced to, or it must not exceed, 5 per cent. per annum. As the interest of L. 1 for one year is so necessary in computations both of simple and compound interest, it may be found, for any rate of interest per cent, by the following proportions:

3	.03	$\frac{1}{33.3}$	3
3.5	.035	$\frac{1}{28.57}$	$3\frac{1}{2}$
4	.04	$\frac{1}{25}$	4
As 100: 4.5, so is 1:	.045 =	$\frac{1}{22.2} =$	$4\frac{1}{2}$ per cent.
5	.05	$\frac{1}{20}$	5
6	.06	$\frac{1}{16.6}$	6
8	.08	$\frac{1}{12.5}$	8
10	.1	$\frac{1}{10}$	10

Interest of L. 1 for one year at

12. In simple interest, let

p , represent any principal lent out upon interest.

r , ——— the interest of L. 1 for one year.

t , ——— the time.

d , ——— any number of days.

i , ——— the interest, and

s , ——— the amount of principal and interest.

Then, $r, 2r, 3r, 4r, \dots tr$, will be the interest of L. 1 for 1, 2, 3, 4, $\dots t$ times; but $1 : tr :: p : trp = i$, the interest of p for t times at the rate r ; and $\frac{drp}{365} =$ interest of p for d days at the rate r . Hence the following

T A B L E.

Given.		
$t, r, p,$	$i = trp = \frac{tp}{20} \text{ or } \frac{tp}{25} \text{ \&c.} \dots$	Equation 1.
$i, t, r,$	$p = \frac{i}{tr} \dots$	2.
$i, r, p,$	$t = \frac{i}{rp} \dots$	3.
$i, t, p,$	$r = \frac{i}{tp} \dots$	4.
$d, r, p,$	$i = \frac{drp}{365} \dots$	5.

EX-

EXAMPLES.

1st, Sought, the interest of L. 356 lent for 12 years, at the rate of $4\frac{1}{2}$ per cent. per annum. Here $trp = 356 \times .54 = 192.24 =$ L. 192, 4s. $9\frac{1}{2}$ d. = interest.

2^d, Sought, the principal whose interest will amount to L. 145 in the space of 9 years, at the rate of 5 per cent. Here $\frac{i}{tr} = \frac{145}{.45} =$ L. 322, 4s. $5\frac{1}{2}$ d. = principal.

3^d, Sought, the time in which L. 567, 10s. will produce L. 306, 9s. of interest, at the rate of 6 per cent. Here $\frac{i}{rp} = \frac{306.45}{34.05} = 9$, the years required.

4th, Sought, the interest of L. 378, 14s. for the space of 127 days, at 5 per cent. Here $\frac{drp}{365} = \frac{2404.7}{365} =$ L. 6; 11:9 = interest required.

13. Again, the interest of any principal p , being manifestly in the compound ratio of p the principal, t the time, and r the rate; it will be, as above, trp ; to which adding p , we shall have s , the amount of principal and interest $= p + trp = 1 + tr \times p$. Hence the following table,

B

Given:

Given.	
$p, t, r,$	$s = p + trp = \overline{1+tr} \times p \dots$ Equation 1.
$s, t, r,$	$p = \frac{s}{1+tr} \dots$ 2.
$s, r, p,$	$t = \frac{s-p}{rp} \dots$ 3.
$s, p, t,$	$r = \frac{s-p}{tp} \dots$ 4.

EXAMPLES.

1st, Sought, the amount of L. 156, lent for 12 years, at the rate of $4\frac{1}{2}$ per cent. Here $\overline{1+tr} \times p = 1.54 \times 156 = 240.24 = \text{L. } 240 : 4 : 9\frac{1}{2} \text{ d.} = s.$

2^d, Sought, the principal which will amount to L. 873, 19 s. in 9 years, at 6 per cent. Here $\frac{s}{1+tr} = \frac{873.95}{1.54} = \text{L. } 567, 10 \text{ s.}$ the principal sought.

3^d, Sought, the time in which L. 600, 14 s. will amount to L. 871.015, at the rate of $4\frac{1}{2}$ per cent. Here $\frac{s-p}{rp} = \frac{270.315}{27.0315} = 10 \text{ years.}$

14. If the principal or interest consists of pounds, shillings, and pence, these parts of a pound may be thrown into a decimal fraction, and

and the interest or amount will be found as above. There is an easy method of converting the decimals of a pound into shillings and pence, viz. double the first decimal on the left hand, calling it so many shillings, to which add 1, if the 2d decimal exceeds 5; prefix the remainder above 5, in the 2d place of decimals, to the decimal in the 3d place, calling them so many farthings, and subtract 1 at or above 25, and 2 at or above 47, and *vice versa*. Thus, L. .7856 = 15 s. 8½ d. and 11 s. 6½ d. = L. .578.

15. If the time *t* consists of years, months, and days, or of days only, their number may be collected from the following table, shewing the number of days from the 1st, 10th, 20th, &c. of one month, to the 1st, 10th, 20th, &c. of any other month; where the months on the left hand column are the former dates, and those at the top are the latter. Thus, from 1st of July to 1st of April following, are 274 days; and from 20th of March to 10th of August, are $153 - 10 = 143$ days.

	1st	10th	20th	31st
1st				
10th				
20th				
31st				
1st				
10th				
20th				
31st				

B 2

TABLE.

TABLE.

To 12, &c. of.

M.	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Jan.	365	31	59	90	120	151	181	212	243	273	304	334
Feb.	334	365	28	59	89	120	150	181	212	242	273	303
March.	306	337	365	31	61	92	122	152	184	214	245	275
April.	275	306	334	365	30	61	91	122	153	183	214	244
May.	245	276	304	335	365	31	61	92	123	153	184	214
June.	214	245	275	304	334	365	30	61	92	122	153	183
July.	184	215	243	274	304	335	365	31	62	92	123	153
August.	153	184	212	243	273	304	334	365	31	61	92	122
Sept.	122	153	181	212	242	273	303	334	365	30	61	91
Oct.	92	123	151	182	212	243	273	304	335	365	31	61
Nov.	61	92	120	151	181	212	242	273	304	334	365	30
Dec.	31	62	90	121	151	182	212	243	274	304	335	365
Days.	31	28	31	30	31	30	31	31	30	31	30	31

From 1st, &c. of.

16. These days may be thrown into the decimals of a year by the following Table.

Days.	Decimals	Days.	Decimals.	Days.	Decimals.
1	.0027397	8	.0219178	60	.1643835
2	.0054794	9	.0246575	70	.1917808
3	.0082192	10	.0273972	80	.2191780
4	.0109589	20	.0547945	90	.2465753
5	.0136986	30	.0821918	100	.2739726
6	.0164383	40	.1095890	200	.5479452
7	.0191780	50	.1369863	300	.8219178

Thus it is required to express 143 days in the decimal fraction of a year.

$$100 = .2739726$$

$$40 = .1095890$$

$$3 = .0082192$$

$$143 = .3917808 \text{ of a year.}$$

Ex. Required the interest of L. 356, 12 s. for 3 years and 143 days, at five per cent. per annum. Here

$$trp = 3.41178 \times .05 \times 356.6 = \text{L. } 60 : 16 : 7\frac{1}{2} = \text{Interest.}$$

17. The interest of any sum of money, for any number of years, at a given rate, may be collected from the following table, calculated for the nine digits, at several rates of simple interest.

T A B L E.

<i>trp</i> for years.						
<i>tp</i>	3 pr cent	3½ pr cent	4 pr cent	4½ pr cent	5 pr cent	6 pr cent
1	.03	.035	.04	.045	.05	.06
2	.06	.07	.08	.09	.10	.12
3	.09	.105	.12	.135	.15	.18
4	.12	.14	.16	.18	.20	.24
5	.15	.175	.20	.225	.25	.30
6	.18	.21	.24	.27	.30	.36
7	.21	.245	.28	.315	.35	.42
8	.24	.28	.32	.36	.40	.48
9	.27	.315	.36	.405	.45	.54

In this table, the decimals, for instance, corresponding to 9 express the interest of L. 9 for 1 year, or of L. 1 for 9 years, or of *p* for *t* years, whatever

whatever these be, provided their product $tp=9$, at a given rate. Of consequence the interest-columns are equal to trp , while the marginal one corresponds to tp . And it is evident that the tabular numbers will not only answer for units, but also for tens, hundreds, or thousands, by removing the decimal point one, two, or three places to the right hand. Hence, in using this table, multiply the principal by the number of years in question, and take out the decimals corresponding to their product; these being properly pointed and added together will express the interest required.

Ex. Required the interest of L. 348, 12 s. for 12 years, at four per cent. per annum. Here $348.6 \times 12 = 4183.2$.

By the Table.

4000	=	160.0
100	=	4.0
80	=	3.2
3	=	.12
.2	=	.008

$$\text{L. } 167:6:6\frac{1}{4} = 167.328 = \text{Interest.}$$

If the given rate be not in the table, the interest may be found by reduction.

18. The interest of any sum of money, for any number of days or months, at a given rate, may be collected from the following table, calculated for the 9 digits, at several rates of simple interest.

TABLE.

TABLE.

$\frac{drp}{365}$ for Days.				$\frac{mrp}{12}$ for Months.		
dp	4 per cent.	$4\frac{1}{2}$ per cent.	5 per cent.	4 pr cent.	$4\frac{1}{2}$ p. cent	5 pr cent.
1	.00010959	.00012328	.000136986	.0033	.00375	.00416
2	.00021918	.00024657	.000273972	.0066	.00750	.00833
3	.00032876	.00036986	.000410958	.0100	.01125	.01250
4	.00043835	.00049315	.00054794	.0133	.01500	.01666
5	.00054794	.00061643	.00068493	.0166	.01875	.02083
6	.00065753	.00073972	.000821918	.0200	.02250	.02500
7	.00076712	.00086301	.00095890	.0233	.02625	.02916
8	.00087671	.00098629	.0010959	.0266	.03000	.03333
9	.00098630	.00110958	.00123287	.0300	.03375	.03750

In this table also, the decimals, for instance, corresponding to 6, express the interest of L. 6 for 1 day, at a given rate, or of L. 1 for 6 days, or of p for d days, whatever these be, provided their product $dp=6$. Of consequence the interest-columns are equal to $\frac{drp}{365}$, while the marginal one corresponds to dp . The same of months. In computing interest by days, which is more usual than by months, the year is always supposed to consist of 365 days; the 29th of February, in leap years, not being taken into the account. Also the day computed from is not reckoned, but the day computed to is: and in discharging bills, three days of grace are allowed the acceptor to make payment.

In using the above table, multiply the principal by the number of days or months in question, and take out the decimals corresponding to their product;

product; these being properly pointed and added together will express the interest required.

Ex. Required the interest of L. 564 for 238 days, at $4\frac{1}{2}$ per cent. per annum. Here $564 \times 238 = 134232$.

By the Table.

100000	=	12.3287
30000	=	3.6986
4000	=	.49315
200	=	.02465
30	=	.00369
2	=	.00024

L. 16, 11 s. = 16.54903 = Interest sought.

If the given rate be not in the table, the interest may be found by reduction; or by the following table.

19. The interest of any sum of money, for any number of days, and at any given rate, may be collected from the following table.

T A B L E.

drp	$\frac{drp}{365}$ for Days.
1	.002739726
2	.005479452
3	.008219178
4	.010958904
5	.013698630
6	.016438356
7	.019178082
8	.021917808
9	.024657534

In this table, the digits in the first column are the products drp ; and the decimals are the quotients of these products divided by 365, and, of consequence, are the interest of p for d days, and at the rate r , whatever these factors be. Therefore to find the interest of any sum of money by the table, multiply

multiply the principal, the number of days, and the rate of L. 1 per annum, into one sum, and from the table take out the interest corresponding to their product.

Ex. Required the interest of L. 342, 15 s. for 260 days, at 5 per cent. Here $342.75 \times 260 \times .05 = 4455.75$.

By the Table.

4000	=	10.9589
400	=	1.09589
50	=	.13698
5	=	.01369
.7	=	.00192
.05	=	.000137

L. 12 : 4 : 2 = 12.207517 = Interest.

20. Or the interest of any sum of money, for any number of days, at five per cent. may be discovered by the following proportion :

$100 \times 365 : 5 :: p \times d : \frac{p \times d}{7300}$. Hence the following equations,

1st, $\frac{p \times d}{7300}$ = Interest of p for d days, }
 2d, $\frac{p}{73}$ = Interest of p for 100 days, } at 5 per
 3d, $\frac{d}{73}$ = Interest of L. 100 for d days, } cent.
 4th, When $\frac{d}{73} = 1, 2, 3, \&c.$ then $\frac{p}{100}$ = interest of p for $73 \times 1, 2, 3, \&c.$ days.

Ex. 1st. Suppose $p = \text{L. } 564$, and $d = 238$ days; then $\frac{p \times d}{7300} = \frac{134232}{7300} = 18.388 = \text{L. } 18 : 7 : 9$, the interest.

C

Ex. 2d,

Ex. 2d, Suppose $p = \text{L. } 342.75$, and $d = 260$ days; then,

$$\frac{p}{73} = \frac{342.75}{73} = 4.6952$$

2

9.3904 = Int. for 200 days.

$$\frac{6}{100} = 2.81712$$

12.20752 = Interest required.

Ex. 3d, Suppose $p = \text{L. } 448$, and $d = 236$ days; then,

$$\frac{d}{73} = \frac{236}{73} = 3.23287 = \text{Interest of L. } 100.$$

4

12.93148

$$\frac{4}{100} = 1.29314$$

$$\frac{1}{3} = .25863$$

14.48325 = Interest of L. 448.

Of Rebate or Discount.

21. To find p , the present value of any sum s , due at any time t , and at any rate of interest r . Here (13, Eq. 2.) $\frac{s}{1+tr} = p$ the present value of s , and $s - p = \text{discount}$.

Ex. What ready money will pay a bill of L. 682, 18s. due 3 years and 9 months hence, discounting interest at the rate of 5 per cent.

Here $\frac{s}{1+tr} = \frac{682.9}{1.1675} = 575.07 = p$, and $682.9 - 575.07 = \text{L. } 107:16:7\frac{1}{4}$ the discount. If the time

time be days, they may be thrown into the decimals of a year by a former table (16.) and then you may proceed as above.

Ex. What is the present value of a bill of L. 285, 10s. due 95 days hence, discounting interest at 5 per cent.

$$95 = .26 = t \quad \frac{285.5}{1.013} = 281.841 = p.$$

$$.05 = r$$

$$1.013 = 1 + tr, \quad 285.5 - 281.841 = 3.66 = \text{discount.}$$

Or thus, seeing $\frac{d}{73} = \text{interest of L } 100 \text{ for } d \text{ days at 5 per cent. (20);}$ hence, in the last example,

$$\frac{95}{73} = 1.3 = \text{Interest of L. } 100 \text{ for 95 days;}$$

$$\text{then } 101.3 : 1.3 :: 285.5 : 3.66 = \text{discount.}$$

$$3.66$$

$$281.84 = \text{present value.}$$

But as the discounting bills in this manner is no better than lending money upon interest, bankers, and others who keep money for the purpose of discounting bills, consider the sum to be paid in the bill as a principal, find the interest thereof for the time to run, adding thereto three days of grace, and deducting the interest so found from the content of the bill, they pay the balance to him for whom the discount is made: some, likewise, for their trouble, charge $\frac{1}{4}$ or $\frac{1}{2}$ per cent. commission, which, before deduction, is added to the interest.

Ex. A bill is presented the 5th of May for discount, of L. 350, 16s. payable 27th of July; how much money does the bearer receive, after

C 2

deduction

deduction of interest and $\frac{1}{4}$ per cent. commission?

Here $p=350.8$, $d=83+3$; hence

$$\frac{350.8 \times 86}{7300} = 4.133 = \text{Interest.}$$

$$.877 = \text{Commission.}$$

$$5.010 = \text{Sum to be deducted.}$$

Hence L. 345.790 paid the bearer of the bill.

But the true discount being precisely equal to the interest of the present value of the bill for the time to run, as calculated in a former example, it is evident that the bearer is a considerable loser by this transaction; and it will be found, that, calling the interest of the bill its discount, the bearer thereof loses exactly, besides commission, the interest of its true discount for the time to run. For calling d the true discount found by 13. Equation 2. then seeing $s=d+p$.

$I=(\text{interest of } s)=\text{interest of } d+p$, for the time of discount; but $d=\text{interest of } p$ for the said time: hence $i-d$, (the bearer's loss when i is the discount)=Interest of d for the time of discount.

Ex. A bill of L. 210 payable a twelvemonth hence is presented to be discounted; required the bearer's loss in this transaction:

L. 10 10 the interest of the bill,

10 00 the true discount.

10 s.=the bearers loss=interest of true discount= $\frac{1}{4}$ per cent. per annum, upon the present worth of the bill, for the time of discount. When the rate of interest is not mentioned, 5 per cent. the legal interest, is always understood.

22. The

22. The several particulars respecting simple interest will more readily be found by the help of the four following tables, wherein are calculated the amount and present value of L. 1 for days or years, at several rates of interest.

TABLE I. Shewing the Amount of L. 1 for Days.

D	4 per cent	4½ per cent.	5 per cent	D	4 pr cent.	4½ pr cent.	5 pr cent.
1	1.00010958	1.00012328	1.00013698	27	1.002959	1.003328	1.003698
2	1.00021917	1.00024657	1.0002739	28	1.003068	1.003452	1.003835
3	1.00032876	1.00036986	1.0004109	29	1.003178	1.003575	1.003972
4	1.00043834	1.00049314	1.0005479	30	1.003287	1.003698	1.004109
5	1.00054794	1.00061643	1.0006849	31	1.003397	1.003822	1.004246
6	1.00065752	1.00073972	1.0008219	32	1.003507	1.003945	1.004383
7	1.00076710	1.00086300	1.0009589	33	1.003616	1.004068	1.004520
8	1.00087668	1.00098628	1.0010958	34	1.003726	1.0041917	1.004657
9	1.00098628	1.0011095	1.0012328	35	1.003835	1.004315	1.004794
10	1.0010958	1.0012328	1.0013698	36	1.003945	1.004438	1.004931
11	1.0012059	1.0013561	1.0015068	40	1.004383	1.004931	1.005479
12	1.0013150	1.0014794	1.0016438	50	1.005479	1.006164	1.006849
13	1.0014246	1.0016027	1.0017808	60	1.006575	1.007397	1.008219
14	1.0015342	1.0017260	1.0019178	70	1.007611	1.008630	1.009589
15	1.0016438	1.0018493	1.0020548	80	1.008767	1.009862	1.010959
16	1.0017534	1.001972	1.0021917	90	1.009863	1.011095	1.012328
17	1.0018629	1.0020958	1.0023287	100	1.010958	1.012328	1.013698
18	1.0019726	1.002219	1.0024657	100	1.021917	1.02465	1.027397
19	1.0020821	1.002242	1.0026027	300	1.03287	1.03697	1.041095
20	1.00219178	1.002465	1.0027397	310	1.03397	1.03822	1.042465
21	1.0023013	1.002589	1.0028767	320	1.03506	1.03945	1.043835
22	1.0024109	1.002712	1.0030137	330	1.03616	1.04068	1.045205
23	1.0025205	1.002835	1.0031506	340	1.03726	1.041917	1.046575
24	1.0026301	1.002958	1.0032876	350	1.03835	1.04315	1.047945
25	1.0027397	1.003082	1.0034246	360	1.03945	1.04438	1.049315
26	1.0028493	1.003205	1.0035616	365	1.04	1.045	1.05

TABLE

TABLE II. Shewing the Amount of L. 1 for Years.

Yrs	4pr cent.	4½p. cen.	5pr cent.	6pr cent.	Yrs	4pr cent.	4½p. cen.	5pr cent.	6pr cent.
1	1.04	1.045	1.05	1.06	31	2.24	2.395	2.55	2.86
2	1.08	1.09	1.10	1.12	32	2.28	2.44	2.60	2.92
3	1.12	1.135	1.15	1.18	33	2.32	2.485	2.65	2.98
4	1.16	1.18	1.20	1.24	34	2.36	2.53	2.70	3.04
5	1.20	1.225	1.25	1.30	35	2.40	2.575	2.75	3.10
6	1.24	1.27	1.30	1.36	36	2.44	2.62	2.80	3.16
7	1.28	1.315	1.35	1.42	37	2.48	2.665	2.85	3.22
8	1.32	1.36	1.40	1.48	38	2.52	2.71	2.90	3.28
9	1.36	1.405	1.45	1.54	39	2.56	2.755	2.95	3.34
10	1.40	1.45	1.50	1.60	40	2.60	2.80	3.00	3.40
11	1.44	1.495	1.55	1.66	41	2.64	2.845	3.05	3.46
12	1.48	1.54	1.60	1.72	42	2.68	2.89	3.10	3.52
13	1.52	1.585	1.65	1.78	43	2.72	2.935	3.15	3.58
14	1.56	1.63	1.70	1.84	44	2.76	2.98	3.20	3.64
15	1.60	1.675	1.75	1.90	45	2.80	3.025	3.25	3.70
16	1.64	1.72	1.80	1.96	46	2.84	3.07	3.30	3.76
17	1.68	1.765	1.85	2.02	47	2.88	3.115	3.35	3.82
18	1.72	1.81	1.90	2.08	48	2.92	3.16	3.40	3.88
19	1.76	1.855	1.95	2.14	49	2.96	3.205	3.45	3.94
20	1.80	1.90	2.00	2.20	50	3.00	3.25	3.50	4.00
21	1.84	1.945	2.05	2.26	51	3.04	3.295	3.55	4.06
22	1.88	1.99	2.10	2.32	52	3.08	3.34	3.60	4.12
23	1.92	2.035	2.15	2.38	53	3.12	3.385	3.65	4.18
24	1.96	2.08	2.20	2.44	54	3.16	3.43	3.70	4.24
25	2.00	2.125	2.25	2.50	55	3.20	3.475	3.75	4.30
26	2.04	2.17	2.30	2.56	56	3.24	3.52	3.80	4.36
27	2.08	2.215	2.35	2.62	57	3.28	3.565	3.85	4.42
28	2.12	2.26	2.40	2.68	58	3.32	3.61	3.90	4.48
29	2.16	2.305	2.45	2.74	59	3.36	3.655	3.95	4.54
30	2.20	2.35	2.50	2.80	60	3.40	3.70	4.00	4.60

Tables I. and II. shew the amount of L. 1 for days or years, and are constructed by Equation 1. 13. whereby n the tabular number $= 1 + tr$. Or thus, Table I. is constructed by taking 364 mean arithmetical proportionals between 0 and the rate of L. 1 for one year, or by dividing this rate by 365, the quotient gives the interest of L. 1 for one day; and this interest added to itself continually gives the interest for any number of days, to which prefixing 1, you have the amount of L. 1 for these days.

days. If a given number of days are not in the table, divide them into such numbers as are to be found there, and to the sum of the decimals corresponding to these numbers prefix 1, the whole will express the amount of L. 1. for the given number of days at a given rate. Or remove the decimal point, where it can be done, one figure to the right hand, and you have the amount for 10 times the tabular number; thus $240 = 1.032876$. at five per cent.—In like manner Table II. is constructed by adding the interest of L. 1 continually to itself for any number of years, to which adding 1, you have the amount of L. 1 for these years. If the given time consists of years and days, to the tabular number corresponding to the years in Table II. add the decimals or interest answering to the days in Table I. their sum will be the amount of L. 1 for the given time. From the nature of these two tables, $1 : n :: p : np = s$ the amount of p , for the time and at the rate corresponding to n . Hence the following table :

TABLE.

Given.			The rate r being given, t the time is found opposite to n ; and t being given, r is found above n in the tables.
$p, t, r,$	$s = np.$		
$s, t, r,$	$p = \frac{s}{n}.$		
$s, r, p, \}$	$\frac{s}{p} = n.$		
$s, t, p, \}$			

Ex. 1st, Sought, the amount, L. 356, 10 s. for 160 days, at 4 per cent. By Table I. $np = 1.017533 \times 356.5 = \text{L. } 362.75$; and interest = L. 6.25

2d, Sought, the amount of L. 300 for 7 years and 165 days, at five per cent. By Table II. and I. $n + n - 1 \times p = 1.3726 \times 300 = \text{L. } 411.78$. amount sought.

3d,

3d, Sought, the time in which L. 300 will amount to L. 400 at 5 per cent. Here $\frac{r}{p} = \frac{4\%}{3\%} = 1.33333 = 6 \text{ years} + 200 + 40 + 3 \text{ days}.$

TABLE III. Shewing the present Value of L. 1 for Days, at five per cent.

Dys		1	2	3	4	5	6	7	8	9
0	.99	9863	9726	9589	9452	9315	9179	9042	8905	8768
10	8632	8495	8359	8227	8086	7950	7815	7678	7541	7403
20	7266	7128	6991	6857	6723	6587	6451	6365	6179	6043
30	5907	5771	5635	5499	5463	5377	5291	5106	4921	4735
40	4550	4414	4279	4144	4009	3873	3738	3603	3468	3332
50	3197	3062	2927	2792	2657	2522	2388	2253	2118	1983
60	1848	1713	1578	1443	1309	1174	1040	0905	0771	0636
70	0502	0367	0233	0099	9965	9831	9697	9563	9430	9296
80	.989163	9028	8893	8758	8623	8489	8356	8223	8090	7956
90	7823	7689	7556	7422	7288	7154	7021	6887	6754	6621
100	6488	6356	6225	6093	5962	5827	5692	5557	5422	5289
110	5156	5023	4891	4758	4625	4490	4360	4227	4094	3961
120	3829	3696	3564	3431	3299	3166	3034	2901	2769	2636
130	2504	2372	2240	2107	1975	1843	1711	1579	1447	1315
140	1184	1052	0920	0788	0657	0525	0394	0262	0131	9999
150	.979868	9736	9605	9473	9342	9210	9079	8947	8816	8684
160	8553	8422	8291	8160	8029	7898	7767	7636	7505	7374
170	7243	7112	6982	6851	6720	6589	6449	6318	6187	6057
180	5936	5806	5675	5545	5415	5284	5154	5024	4893	4763
190	4633	4503	4373	4243	4113	3983	3853	3723	3593	3463
200	3333	3203	3074	2944	2815	2685	2556	2426	2297	2167
210	2038	1909	1779	1650	1521	1391	1262	1133	1003	0874
220	0745	0616	0487	0358	0229	0101	9972	9843	9714	9585
230	.969457	9328	9200	9071	8942	8814	8685	8557	8428	8309
240	8171	8043	7914	7786	7658	7529	7401	7273	7144	7016
250	6888	6760	6632	6504	6376	6248	6121	5993	5865	5737
260	5609	5481	5354	5226	5098	4971	4843	4716	4588	4460
270	4333	4205	4078	3951	3824	3697	3570	3442	3315	3188
280	3061	2934	2807	2680	2553	2426	2300	2173	2046	1919
290	1792	1665	1539	1412	1286	1159	1033	0906	0770	0653
300	0527	0401	0274	0148	0022	9896	9769	9643	9517	9391
310	.959265	9139	9013	8887	8761	8635	8509	8384	8258	8132
320	8006	7880	7755	7629	7503	7378	7252	7127	7001	6875
330	6750	6625	6499	6374	6249	6124	5999	5873	5748	5623
340	5498	5373	5248	5123	4998	4873	4748	4624	4499	4374
350	4249	4124	3999	3875	3750	3625	3501	3376	3251	3127
360	3002	2878	2753	2629	2505	2381				

TABLE

TABLE IV. Shewing the present Value of
L. 1 for Years.

Yrs	4 pr cent.	5 pr cent.	6 pr cent.	Yrs	4 pr cent.	5 pr cent.	6 pr cent.
1	.961538	.952381	.943396	26	.490196	.434781	.390625
2	.925926	.909091	.892857	27	.480769	.425532	.381679
3	.892857	.869565	.847457	28	.471698	.416666	.373134
4	.862069	.833333	.806451	29	.462963	.408163	.364963
5	.833333	.8	.769230	30	.454545	.4	.357143
6	.806451	.769230	.735294	31	.446428	.393157	.349650
7	.781250	.740740	.704225	32	.438596	.384615	.342466
8	.757576	.714286	.675675	33	.431034	.377358	.335570
9	.735294	.689655	.649350	34	.422728	.370370	.328947
10	.714286	.666666	.625	35	.416666	.363636	.322581
11	.694444	.645161	.602409	36	.409836	.357143	.316456
12	.675675	.625	.581395	37	.403226	.350877	.310559
13	.657895	.606060	.561797	38	.396825	.344827	.304878
14	.641025	.588235	.543478	39	.390625	.338983	.299401
15	.625	.571428	.526315	40	.384615	.333333	.294117
16	.609756	.555555	.510204	41	.378788	.327870	.289017
17	.595238	.540540	.495049	42	.373134	.322581	.284091
18	.581395	.526315	.480769	43	.367647	.317460	.279330
19	.568182	.512820	.467289	44	.362319	.3125	.274725
20	.555555	.5	.454545	45	.357143	.307692	.270270
21	.543478	.487804	.442477	46	.352113	.303030	.265957
22	.531915	.476190	.431034	47	.347222	.298509	.262383
23	.520833	.465116	.420168	48	.342466	.294117	.257732
24	.510204	.454545	.409836	49	.337838	.289855	.253807
25	.5	.444444	.4	50	.333333	.285714	.25

Tables III. and IV. shew the present value of L. 1 for days or years, and are constructed by Equation 2d, 13. whereby n , the tabular number, $= \frac{1}{1+tr}$; or they are the reciprocals of Table I. and II. whereby $\frac{1}{n}$ of Table I. and II. $= n$ of Table III. and IV. From the nature of these two Tables, $1 : n :: s : ns = p$, the principal or present worth of s , for the time and at the rate corresponding to n .

D

When

When the time of discount consists of years and days, in this case, $n \times ns = p$, the present worth. Hence the following

TABLE.

Given.	
$s, t, r,$	$p = ns.$
$p, t, r,$	$s = \frac{p}{n}$
$s, r, p, \}$ $s, t, p, \}$	$n = \frac{p}{s}$

Ex. 1st, What ready money will take up a bill of L. 685, 12 s. due 255 days hence, interest being reckoned at 5 per cent? By Table III $ns = .966248 \times 685.6 = \text{L. } 662.4596 = p$, and discount = L. 23.14.

2d, Required the present worth of L. 250, discounting for 10 years, at the rate of 5 per cent. By Table IV, $ns = .6666 \times 250 = \text{L. } 166.666 = p$, and discount = L. 83.33.

3d, What ready money will pay a debt of L. 150 due 5 years and 95 days hence, interest being reckoned at 5 per cent.? By Table IV. and III. $ns = .8 \times .987154 \times 150 = \text{L. } 118.45848 = p$, and L. 31.541 = discount.

4th, Having s, p , and r , in the last example, sought, t the time. $\frac{p}{s} = \frac{118.45848}{150} = .7897232 = nn$, and $\frac{.7897232}{.8 = 5 \text{ yrs}} = .987154 = 95 \text{ days}$.

These examples may with equal facility be determined by the equations annexed to Table I. and II.

Re-

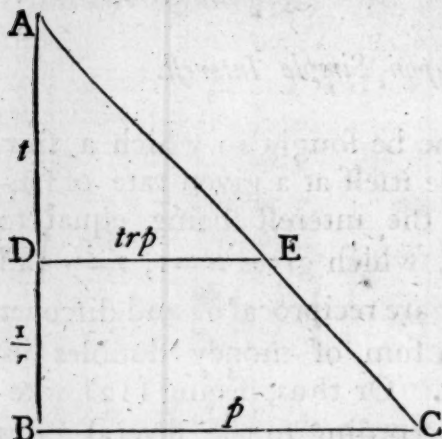
Remarks upon Simple Interest.

23. 1st, If the time be sought in which a sum of money will double itself at a given rate of interest. In this case, the interest being equal to the principal, $trp=p$, which gives $tr=1$, $t=\frac{1}{r}$ and $r=\frac{1}{t}$; so that t and r are reciprocal of and discover one another, when a sum of money doubles itself at simple interest. Or thus, seeing (12) interest $=\frac{p}{20}$, or $\frac{p}{25}$, &c. according to the several rates of interest; hence if $t=20$, 25, &c. when the rates are 5, 4, &c. per cent. the interest will equal the principal. So that

Interest= p , when $t=\frac{1}{r}=$	33.33	3
	28.57	$3\frac{1}{2}$
	25	4
	22.27 years at	$4\frac{1}{2}$ per cent. per ann.
	20	5
	16.66	6
	12.5	8
	10	10, &c.

2d, The simple interest of any sum of money is, *ceteris paribus*, in the direct proportion of p , t , or r ; so that doubling p , while t and r remain the same, the interest will be doubled also, &c. For as interest $=trp$, if you increase or diminish any one of these three factors, the interest will be increased or diminished in the same proportion.

3d, The interest of any sum of money, for a given time, and at a given rate, may be discovered by means of a right-angled triangle. Thus.



In constructing the right-angled triangle ABC , make $AB = \frac{1}{r}$, and $BC = p$; in AB take the point D , above or below

BC , making $AD = t$, and join AC ; then DE drawn parallel to BC will represent the interest sought. For AB , or $\frac{1}{r}$ being equal to the time in which p will double itself at the rate r , $BC = \text{interest of } p \text{ at the end of that time}$; but $AB : BC :: AD : DE$, therefore, by the last remark, $DE = \text{interest of } p \text{ for the time } t$, and at the rate r .

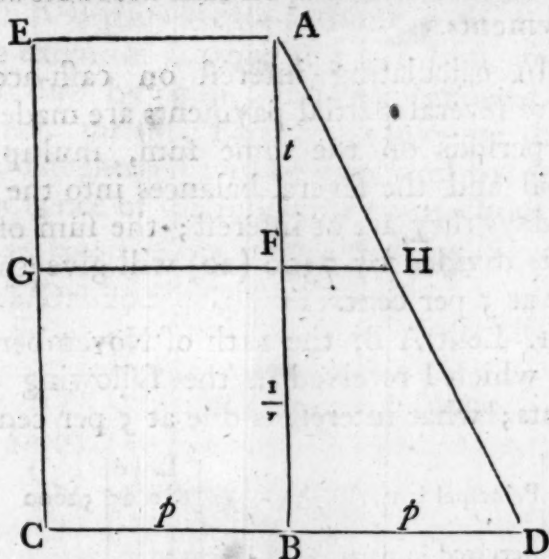
Ex. Suppose $p = \text{L. } 460$, $t = 12$ years, and the rate 5 per cent.; then $AB = 20$, $BC = 460$, $AD = 12$, and $DE = \text{L. } 276 = \text{interest}$.

4th, The different amounts are, *ceteris paribus*, in the direct proportion of their principals or present worths; so that doubling p , while t and r remain the same, the amount s will be doubled also, &c. For as the amount $s = p + trp$, so $2s = 2p + tr \times 2p$.

But the different amounts of the same principal, or the different present worths of the same bill, are not in the direct proportion of t or r ; for $2s$ is greater than $p + 2trp$; and though $\frac{s}{1+tr} = p$, yet

yet $\frac{s}{1+2tr}$ is greater than $\frac{1}{2}p$. Hence the different discounts of the same bill are not in the direct proportion of the times or rates.

5th, The amount of any sum of money, for a given time, and at a given rate, may be discovered by a geometrical figure, two of whose opposite sides are parallel, and two of its angles right. Thus, in constructing the annexed figure, make



$AB = \frac{1}{r}$, CD , bisected in B , $= 2p$; in AB take $AF = t$; and having completed the parallelogram AC and triangle ABD , GH drawn through F parallel to CD will represent the amount sought. For as $GF = p$, and FH , by remark 3d, is equal to trp , hence $GH = s$ the amount.

Ex. Suppose $p = \text{L. } 300$, $t = 9$ years, and the rate 5 per cent.; then $AB = 20$, $AF = 9$, $CD = 600$, $GF = 300$, and $GH = \text{L. } 435$, the amount.

6th, As in simple interest, strictly speaking, the interest is supposed not to be uplifted while the

the principal remains in the borrower's hands; it must alter the case greatly when that interest is paid yearly, and make it partake of the nature of compound interest, seeing one hath the use, and can dispose of that interest.

Supplement to Chapter II.

1st, Of interest due upon cash-accounts and partial payments.

24. In calculating interest on cash-accounts, or where several partial payments are made at different periods on the same sum, multiply the principal and the several balances into the number of days they are at interest; the sum of these products divided by 7300 (20) will give the total interest at 5 per cent.

Ex. 1. Lent A B, the 11th of November 1788, L. 800, which I received in the following partial payments; what interest is due at 5 per cent?

		L.	d	
1788.	Nov. 11. Principal lent, -	800	65	52000
1789.	Jan. 15. Received in part, -	250		
	Balance,	550	92	50600
	April 17. Received in part, -	150		
	Balance,	400	62	24800
	June 18. Received in part, -	100		
	Balance,	300	62	18600
	Aug. 19. Received in part, -	125		
	Balance,	175	37	6475
	Sept. 25. Received in full of principal,	175		
		0		1524.75 = 20.887
				= Int.

Bankers sometimes receive at 4 and lend at 5 per cent. In this case it may be proper to consider the money lent by the banker as debtor, and the money received by him as creditor, and to make two columns for products, where the interest arising from the debtor products is to be computed at five, and the other at 4 per cent. ; the difference of these two is the balance of interest due to or by the banker.

Ex. 2. A B runs a cash-account with a banker to the extent of L. 500, at 5 per cent. for the balances due by him, and has allowed him 4 per cent. for such balances as may be due to him. What interest will be due, and by whom, in consequence of the following transactions?

1789.		L.	d.	Dr	Cr
Jan. 8.	Dr	200	61	12200	
March 10.	Cr	350			
	Cr	150	66		9900
May 15.	Dr	310			
	Dr	160	66	10560	
July 20.	Cr	240			
	Cr	80	52		4160
Sept. 10.	Dr	410			
	Dr	330	52	17160	
Nov. 1.	Cr	420			
	Cr	90	49		4410
Dec. 20.	Dr	250			
1790.					
Jan. 8.	Dr	160	19	3040	
				42960	18470

$$\frac{429.60}{73} - \frac{4}{5} \left) \frac{184.7}{73} = \text{L. } 3.86 \text{ Int. } \right\} \text{ due to the banker.}$$

$$160.00 \text{ Prin.}$$

When partial payments are made on bills or bonds at any interval greater than a year, the legal and usual method is to add the interest at the times of payment to the principal, and from that amount to deduct the payment.

Ex. 3d. Borrowed on bond, June 1. 1781, the sum of L. 1000, at 5 per cent. and made partial payments as follows; required, the state of the affair on the 14th of August 1785.

		L.
1781.		
June 1.	Principal borrowed,	1000
	Interest for 1 year and 129 days, -	67.671
1782.	Amount,	1067.671
Oct. 8.	Paid in part, -	250
	Balance,	817.671
	Interest for 1 year and 85 days, -	50.4
1784.	Amount,	868.071
Jan. 1.	Paid in part, -	408.071
	Balance,	460.
	Interest for 1 year and 225 days, -	37.178
1785.	Amount,	497.178
Aug. 14.	Paid in full, -	497.178
		2. Of

2d. Of the equation of payment.

25. When two or more debts are payable at different times; the finding a mean time at which all the debts may be paid at once, without loss to debtor or creditor, is called equating the terms of payment, and may be performed by the following

R U L E.

Multiply the several debts into their respective times, and divide the sum of the products by the total debts; the quotient is accounted the mean time.

Ex. A owes B L.800, whereof L.200 are to be paid at the end of 2 months, L. 300 at the end of 6 months, and L. 300 at the end of 10 months; required the equated time for paying the whole.

$$200 \times 2 = 400$$

$$300 \times 6 = 1800$$

$$300 \times 10 = 3000$$

$$\hline 800$$

$$\hline 5200 (6.5 \text{ months, answer.})$$

The above method is easy, but not accurate: for a person, by keeping money unpaid after it becomes due, gains the interest thereof for that time; but by paying money before it is due, he does not lose the interest, as the Rule supposes, but only the discount thereof for that time, which is always less than the interest, by the interest of the discount for the time; therefore take the following

R U L E.

Find, by Equation 2d, 13. the present worth of each debt, and by Equation 3d, find in what time the sum of the present worths will amount to the sum of the debts.

E

Ex. A

Ex. A owes B L. 250, whereof L. 50 is payable 1 year hence, L. 100 3 years hence, and L. 100 5 years hence; what is the equated time for paying all these debts at once, interest reckoned at 5 per cent.? Here,

$$\frac{50}{1.05} = 47.619 \quad \text{then} \quad \frac{s-p}{rp} = \frac{35.475}{10.728} = 3.302 = 3 \text{ years}$$

$$\frac{100}{1.15} = 86.956 \quad \text{and 110 days.}$$

$$\frac{100}{1.25} = \frac{80.000}{214.575} = p.$$

3d. Of the premium of commission.

It is usual with bankers, in granting a draught upon their correspondents either at home or abroad, payable to the bearer or order, for value received, to charge so much per cent. upon the transaction, in name of commission; which commission varies from $\frac{1}{8}$ to $2\frac{1}{2}$ per cent. according to the nature of the case; hence the following

T A B L E.

Per cent.			Com. on L. 1.	Per cent.			Com. on L. 1.
L.	s.	d.		L.	s.	d.	
$\frac{1}{8} = 0$	2	6	.00125	$1\frac{1}{4} = 1$	5		.0125
0	3		.0015	$1\frac{1}{2} = 1$	10		.015
$\frac{1}{6} = 0$	3	4	.00166	$1\frac{3}{4} = 1$	15		.0175
$\frac{1}{5} = 0$	4		.002	2	2	0	.02
$\frac{1}{4} = 0$	5		.0025	$2\frac{1}{4} = 2$	5		.0225
$\frac{1}{3} = 0$	6	8	.0033	$2\frac{1}{2} = 2$	10		.025
$\frac{1}{2} = 0$	10		.005	$2\frac{3}{4} = 2$	15		.0275
$\frac{3}{4} = 0$	15		.0075	3	3	0	.03
1	1	0	.01	$3\frac{1}{2} = 3$	10		.035

Ex. Sought, the commission upon L. 542 at $1\frac{1}{2}$ per cent. Here $542 \times .015 = \text{L. } 8.13$, the commission.

C H A P. III.

Of ANNUITIES computed at SIMPLE INTEREST.

26. **A**N annuity, rent, or pension, is a sum of money payable yearly, every half year, or quarterly, to continue for a certain number of years, for life, or for ever. When an annuity continues unpaid after it falls due, it is then said to be in arrears. When the purchaser does not immediately enter upon possession, the annuity not commencing till some time after, it is said to be in reversion; and the reversion of an annuity is the present value of a perpetuity which is to commence after the expiration of the annuity, being the same with a reversion in fee-simple.

1st. Of the amount of an annuity in arrears.

27. In computing annuities at simple interest, let,

a represent the annuity, rent, or pension.

p its present value.

t the time of its continuance.

n the number of years before it commences.

r the interest of L. 1 for one year.

s the amount of an annuity and its interest.

E 2

Then,

Then, since $L. 1:r::a:ar$, and as the first year's annuity bears no interest, being due only at the end of the year, we shall have, from an yearly annuity, a ,

$$a \times \frac{1+0}{1} = \text{amount of the 1st year.}$$

$$a \times \frac{1+r}{1} = \text{amount of the 2d year.}$$

$$a \times \frac{1+2r}{1} = \text{amount of the 3d year.}$$

$$a \times \frac{1+3r}{1} = \text{amount of the 4th year.}$$

$$a \times \frac{1+t-1 \times r}{1} = \text{amount of the } t \text{ year.}$$

Here the number of terms of this series being t , the sum $s = ta \times 1 + \frac{t-1}{2}r$ (8). Hence the following

T A B L E.

Given.	
$a, t, r,$	$s = t a \times 1 + \frac{t-1}{2}r = ta + \frac{t}{2} \times tra - ra.$ Equation 1.
$s, t, r,$	$a = \frac{s}{t \times 1 + \frac{t-1}{2}r},$ 2.
$s, t, a,$	$r = \frac{\frac{2s}{t} - 2a}{t-1 \times a}$ 3.
$s, r, a,$	$t = \sqrt{\frac{2s}{ar} + \frac{1}{r} - \frac{1}{2} - \frac{1}{r} - \frac{1}{2}}$ 4.

Sought r , in Equation 3d.

$$\begin{array}{l} \text{Equ. 1.} \left| \begin{array}{l} s = ta + \frac{t^2 a - ta}{2} r. \\ 2s - 2ta = t^2 a - ta \times r. \\ \frac{2s}{t} - 2a = t - 1 \times ar. \end{array} \right.$$

Sought t , in Equation 4th.

$$\begin{array}{l} \text{Equ. 1.} \left| \begin{array}{l} 2s = 2at + \frac{t^2 a - ta}{2} \times r. \\ \frac{2s}{r} = \frac{2at}{r} + t^2 a - ta. \\ t^2 + \frac{2t}{r} - t = \frac{2s}{ar}. \\ t + \frac{1}{r} - \frac{1}{2} = \sqrt{\frac{2s}{ar} + \frac{1}{r} - \frac{1}{2}} \end{array} \right.$$

Ex. 1st.

Ex. 1st, Required the amount of an annuity of L. 20 per annum, being in arrears for 12 years, at 5 per cent. simple interest. Here $a=20$, $t=12$, $r=.05$, and $ta \times \frac{1+\frac{t-1}{2}r}{2} = 240 \times 1.275 = 306 = s$.

2d, Sought, the annuity, which being in arrears for 9 years will amount to L. 250, at 5 per cent. Here $s=250$, $t=9$, $r=.05$.

$$\frac{s}{t \times 1 + \frac{t-1}{2}r} = \frac{250}{10.8} = \text{L. } 23.148 = \text{annuity.}$$

3d, Sought, the rate of interest at which an annuity of L. 20 per annum, being in arrears for 12 years, will amount to L. 306.

$$\frac{\frac{2s}{t} - 2a}{s - 1 \times a} = \frac{11}{220} = .05 = 5 \text{ per cent.}$$

4th, In what time will an yearly annuity of L. 85.5 amount to L. 2522.25, at the rate of 5 per cent. per annum?

$$\sqrt{\frac{2s}{ar} + \frac{1}{r} - \frac{1}{2}} - \frac{1}{r} - \frac{1}{2} = \sqrt{1180 + 380.25} - 19.5 = 20 \text{ years.}$$

28. When an annuity is paid every half year, quarterly, or monthly; then for half years payments, take half the yearly annuity, half of the ratio, and twice the number of years; and for quarterly payments, take the fourth part of the yearly annuity and of the ratio, and four times the number of years, &c. and work as directed above.

To Extract the Square Root of any Quantity.

Let the given quantity = g , and its square root = $r + e$, then
 $rr + 2re + ee = g$
 $2re + ee = g - rr = d$
 $\frac{d}{2r+e} = e$

Example.

$$\begin{array}{l} \text{Let } g = 1560.25 (39.5 = r + e) \\ rr = 9 \\ \hline 2r + e = 69 \overline{) 660 = d} \\ \underline{621} \\ 2r + e = 785 \overline{) 3925 = d} \\ \underline{3925} \end{array}$$

Ex. What

Ex. What will an yearly annuity of L. 70, payable every quarter of a year, amount to in 5 years, at 5 per cent. Here $a=17.5$, $t=20$, $r=.0125$, and $350 \times 1.11875 = 391.56 = s$.

2d, Of the present worth of an annuity.

29. When an annuity is to continue for a certain number of years, its present value p , at a given rate of simple interest, may be found thus: Since $1+tr$, the amount of L. 1 for any time, hath the same ratio to L. 1, as $ta \times 1 + \frac{t-1}{2}r$, the amount of any annuity a , hath to p its present value; therefore $1+tr \times p = ta \times 1 + \frac{t-1}{2}r$, from which all the varieties in this case will be resolved, as in the following

T A B L E.

Given.		
$a, r, t,$	$p = ta \times \frac{1 + \frac{t-1}{2}r}{1+tr}$	Equ. 1.
$p, r, t,$	$a = \frac{p}{t} \times \frac{1+tr}{1 + \frac{t-1}{2}r}$	2.
$p, t, a,$	$r = \frac{\frac{p}{t} - a}{\frac{t-1}{2}a - p}$	3.
$p, r, a,$	$t = \sqrt{\frac{2p}{ar} + \frac{1}{r} - \frac{p}{a} - \frac{1}{2} - \frac{1}{r} - \frac{p}{a} - \frac{1}{2}}$	4.

Sought r in Equation 3d.

Equ. 1. $p + trp = ta + \frac{at^2 - at}{2}r.$

$$\frac{p}{t} + rp = a + \frac{t-1}{2}ar.$$

$$\frac{p}{t} - a = \frac{t-1}{2}ar - rp.$$

Sought t in Equation 4.

Equ. 1. $2p + 2rpt = 2at + t^2 - 1 \times ar.$

$$\frac{2p}{ar} + \frac{2pt}{a} = \frac{2}{r}t + t^2 - t.$$

$$\left. \begin{array}{l} + \frac{2}{r} \\ t^2 - \frac{2p}{a} \\ - 1 \end{array} \right\} t = \frac{2p}{ar}$$

Ex. 1st.

Ex. 1. Required the present value of a yearly annuity of L. 20, to continue 12 years, allowing discount at 5 per cent. simple interest. Here,

$$ta \times \frac{1 + \frac{r-1}{2}r}{1+r} = 240 \times \frac{1.275}{1.6} = \text{L. } 191.2488.$$

2. A person is willing to lay out the sum of L. 300 in the purchase of an yearly annuity, to continue 15 years, being allowed discount at 6 per cent; sought the annuity. Here $\frac{p}{t} \times \frac{1+r}{1+\frac{r-1}{2}r} = 20 \times 1.338 = \text{L. } 26.76 = \text{annuity.}$

3. There is an annuity of L. 50 a-year, to continue 20 years, to be sold for L. 737.5, ready money; sought, the rate of simple interest at which the discount is given.

$$\frac{\frac{p}{t} - a}{\frac{1-r}{2} - p} = \frac{-13.125}{-262.5} = .05 = 5 \text{ per cent.}$$

4. There is an yearly annuity of L. 30, to continue a certain number of years, which can be sold for L. 405, discounting at 5 per cent; sought, the time of its continuance. $\sqrt{540+36}-6=18$ years.

30. When an annuity is paid every half year, quarterly, or monthly, and its present value be sought. See 28.

Ex. What will be the present value of an yearly annuity of L. 70, payable every quarter of a year, and to continue 5 years, discounting interest at 5 per cent. Here $a=17.5$, $t=20$, $r=.0125$, and $350 \times .895 = 313.25 = p$.

In like manner a , t , or r may be found.

3d, Of

3d. Of the present worth of an annuity in reversion.

31. When an annuity is to commence after a certain number of years (n), and then to continue for a certain time (t), its present value may be found thus: Find its amount at the end of the time t , (Equation 1. 27.) and divide this amount by $1+t+nr$, the quotient will give p , the present value. Thus $ta \times \frac{1 + \frac{t-1}{2}r}{1+t+nr} = p$.

Ex. Required the present value of an yearly annuity of L. 30, which is to commence 6 years hence and to continue 15 years, discounting interest at 5 per cent. Here $a=30$, $t=15$, $n=6$, $r=.05$, and $450 \times \frac{1.35}{2.05} = \text{L. } 296.3 = p$.

4th. Of the present worth of the reversion of an annuity.

32. The present value of the reversion of an annuity, being the same with that of a perpetuity which is to commence after a certain number of years (n), may be found thus: From the value of the perpetuity (see Remark 3d), subtract that of the annuity, the remainder gives the present value of the reversion: Or thus, divide the value of the perpetuity, viz. $\frac{a}{r}$ by $1+nr$, the quotient gives the present value of the reversion, thus $\frac{\frac{a}{r}}{1+nr} = p$.

Ex. Sought, the reversion of an yearly annuity of L. 20, which is to continue 12 years, allowing discount at 5 per cent. Here

$$\frac{\frac{a}{r}}{1+nr} = \frac{20}{.05} = \text{L. } 250 \text{ the reversion.}$$

33. The

33. The several particulars respecting annuities at simple interest may be discovered by means of the two following Tables, wherein are calculated the amount and present value of L. 1 annuity for years, at several rates of interest.

TABLE V. Shews the Amount of L. 1 Annuity for Years.

Yrs	4 pr cent.	5 pr cent.	6 pr cent.	Yrs	4 pr cent.	5 pr cent.	6 pr cent.
1	1.00	1.00	1.00	26	39.00	42.35	45.50
2	2.04	2.05	2.06	27	41.04	44.55	48.06
3	3.12	3.15	3.18	28	43.12	46.90	50.68
4	4.24	4.30	4.36	29	45.24	49.30	53.36
5	5.40	5.50	5.60	30	47.40	51.75	56.10
6	6.60	6.75	6.90	31	49.60	54.25	58.90
7	7.84	8.05	8.26	32	51.84	56.80	61.76
8	9.12	9.40	9.68	33	54.12	59.40	64.68
9	10.44	10.80	11.16	34	56.44	62.05	67.66
10	11.80	12.25	12.70	35	58.80	64.75	70.70
11	13.20	13.75	14.30	36	61.20	67.50	73.80
12	14.64	15.30	15.96	37	63.64	70.30	76.96
13	16.12	16.90	17.68	38	66.12	73.15	80.18
14	17.64	18.55	19.46	39	68.64	76.05	83.46
15	19.20	20.25	21.30	40	71.20	79.00	86.80
16	20.80	22.00	23.20	41	73.80	82.00	90.20
17	22.44	23.80	25.16	42	76.44	85.05	93.66
18	24.12	25.65	27.18	43	79.12	88.15	97.18
19	25.84	27.55	29.26	44	81.84	91.30	100.76
20	27.60	29.50	31.40	45	84.60	94.50	104.40
21	29.40	31.50	33.60	46	87.40	97.75	108.10
22	31.24	33.55	35.86	47	90.24	101.05	111.86
23	33.12	35.65	38.18	48	93.12	104.40	115.68
24	35.04	37.80	40.56	49	96.04	107.80	119.56
25	37.00	40.00	43.00	50	99.00	111.25	123.50

Table V. shews the amount of L. 1 annuity, and is constructed by equation 1. 27. whereby n the tabular number $= t \times 1 + \frac{t-1}{2} r$; Or thus, seeing

$1 = a$ the amount for 1 year,

$a + 1 + r = b$ for 2 years,

$b + 1 + 2r = c$ for 3 years,

$c + 1 + 3r = d$ for 4 years, &c.

F

Hence,

Hence, to L 1, the first year of this Table, add the first year of Table II. their sum is the second year of this Table; to which add the second year of Table II. their sum is the third year of this Table, &c. From the nature of the Table, $1:n::a:na=s$ the amount of any annuity a , for the time and at the rate corresponding to n : hence the following

TABLE.

Given.	
$a, t, r,$	$s = na,$
$s, t, r,$	$a = \frac{s}{n}.$
$s, t, a, \}$	$n = \frac{s}{a}.$
$s, r, a, \}$	

Ex. 1. what will an yearly annuity of L. 35 amount to in 16 years, at the rate of 5 per cent. simple interest? $na = 22 \times 35 = \text{L. } 770$, the amount.

2. What yearly annuity will amount to L. 471 in the space of 20 years, at 6 per cent. simple interest? $\frac{s}{n} = \frac{471}{31.4} = \text{L. } 15$, the annuity.

TABLE

TABLE VI. Shews the present Value of L. 1 Annuity for Years.

Ys	4 per cent.	5 per cent.	6 per cent.	Ys	4 per cent.	5 per cent.	6 per cent.
1	0.96154	0.95238	0.94339	26	19.11764	18.36956	17.77343
2	1.88888	1.86363	1.83928	27	19.71154	18.95744	18.34351
3	2.78571	2.73913	2.69491	28	20.33962	19.54166	18.91044
4	3.65517	3.58333	3.51613	29	20.92592	20.12245	19.47262
5	4.5	4.4	4.30769	30	21.54545	20.7	20.03571
6	5.32258	5.19231	5.07353	31	22.14285	21.27451	20.59440
7	6.125	5.96296	5.81690	32	22.73688	21.84615	21.15067
8	6.90909	6.71428	6.54054	33	23.32758	22.41509	21.70469
9	7.67647	7.44827	7.24675	34	23.91525	22.98148	22.35526
10	8.42857	8.16666	7.9375	35	24.5	23.54545	22.80645
11	9.16666	8.87096	8.61445	36	25.08197	24.10714	23.35443
12	9.89189	9.5625	9.27907	37	25.66129	24.66666	23.90062
13	10.60526	10.24242	9.93258	38	26.23809	25.22414	24.44512
14	11.30769	10.91176	10.58326	39	26.8125	25.77965	24.98802
15	12.	11.57143	11.21052	40	27.38461	26.33333	25.52941
16	12.68392	12.22222	11.83673	41	27.95454	26.88524	26.06936
17	13.35714	12.86486	12.45544	42	28.52238	27.43548	26.60795
18	14.02325	13.5	13.06706	43	29.08823	27.98412	27.14525
19	14.68182	14.12820	13.67289	44	29.65217	28.53125	27.68132
20	15.33333	14.75	14.27272	45	30.21428	29.07692	28.21621
21	15.97826	15.36585	14.86504	46	30.77465	29.62121	28.75
22	16.61702	15.97619	15.45689	47	31.33333	30.16418	29.28272
23	17.24948	16.58139	16.04201	48	31.89041	30.70588	29.81444
24	17.87755	17.18182	16.60246	49	32.44594	31.24638	30.34518
25	18.5	17.77777	17.2	50	33.	31.78571	30.875

Table VI. shews the present value of L. 1 annuity, and is constructed by Equation 1st, 29.

whereby n the tabular number $= t \times \frac{1 + \frac{t-1}{2} r}{1 + tr}$; Or thus, n Table VI. $= \frac{n, T. V.}{n, T. II.}$.

This Table is constructed upon the supposition, that the present value of an annuity is equal to that of the amount of the several annual payments, with their interest at the expiration of the annuity. From the nature of the Table,

$1 : n :: a : na = p$, the present worth of any annuity a , for the time, and at the rate corresponding to n . Hence the following

T A B L E.

Given.	
$a, r, t,$	$p = na.$
$p, r, t,$	$a = \frac{p}{n}.$
$p, t, a, \}$	$n = \frac{p}{a}.$
$p, r, a, \}$	

Ex. 1st. What is the present worth of an yearly annuity of L. 50, to continue 15 years, discounting interest at 5 per cent?

$$na = 11.57143 \times 50 = \text{L. } 578.5715 = p.$$

2d. A person is willing to lay out the sum of L. 300 in the purchase of an annuity to continue 18 years, being allowed discount at 5 per cent.; sought the annuity.

$$\frac{p}{n} = \frac{300}{13.3} = \text{L. } 22.222 = \text{annuity.}$$

3d. A gentleman hath L. 160, which he would lay out in the purchase of an annuity of L. 20 per annum; sought, how many years the said annuity must continue, discounting interest at 6 per cent.

$\frac{p}{a} = \frac{160}{20} = 8 = n$. the next less to which below 6 per cent. is 7.9375, corresponding to 10 years, and for the odd days say

$$\begin{array}{r} 8.61445 \\ 7.93750 \\ \hline \end{array} \quad \begin{array}{r} 8.0000 \\ 7.9375 \\ \hline \end{array}$$

as .67695 : 1 :: .0625 : .09233, corresponding to 34 days, (16).

4th. A

4th. A hath an annuity of L. 20 per annum, to continue 7 years; B hath one of L. 5, 10 s. to continue 21 years; sought, which of these is most valuable, discounting interest at 6 per cent.

$$\text{A's annuity} = 5.8169 \times 20 = 116.3380$$

$$\text{B's annuity} = 14.865 \times 5.5 = 81.7575$$

$$\text{A} - \text{B} = \text{L. } 34.5805$$

5th. A lends to B L. 360, upon a mortgage of land, whose rent is L. 75 per annum; B keeps the money 5 years, during which time A receives the said rent; sought, the state of their affairs, discounting interest at 6 per cent.

$$\text{By Table II. } 360 \times 1.3 = \text{L. } 468$$

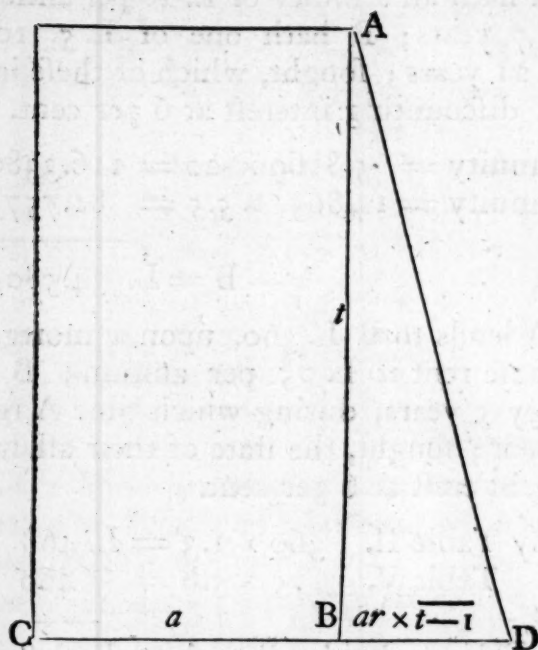
$$\text{Table V. } 75 \times 5.6 = 420$$

$$\text{Balance due by B to A} = \text{L. } 48$$

Remarks upon Annuities at Simple Interest.

34. 1st, The amount of an annuity for a given time, and at a given rate of simple interest, may be expressed by the area of a rectangle and triangle.

Thus,



Thus, in constructing the annexed figure, make $AB=t$, BC , raised at right angles to AB , $=a$, and $BD = \text{interest of } a \text{ for the time } t-1$; lastly, join AD , and complete the rectangle AC ; then will $AC + ABD$ express the amount of the annuity a , for the time t , at a given rate of interest; for $AC + ABD = at + \frac{t}{2} \times tra - ra = \text{amount of the annuity}$.

Ex. Suppose $a = \text{L. } 20$, $t = 12$ years, and the rate 5 per cent.; sought the amount.

Here $AB = 12$, $BC = 20$, $BD = 11$. Hence, $AC + ABD = 240 + 66 = \text{L. } 306 = \text{amount}$.

2d. The amount of an annuity is, *cæteris paribus*, always less, and its present value greater, upon the principles of simple than of compound interest, as is manifest from inspecting the Tables; and

and as the tabular numbers expressing the present worth of an annuity may be considered as the years purchase of said annuity for the corresponding years, these at simple interest are unlimited. Thus an annuity to continue 99 years, at five per cent. is valued, Equation 1st, 29. at 57.4 years purchase; whereas the years purchase at compound interest are always less than $\frac{1}{r}$. For these reasons the purchase of annuities upon the principles of simple interest is very unequitable, and compound interest, that both parties may employ their money to the greatest advantage, is usually admitted in these transactions.

3d. The value of a perpetuity cannot be obtained upon the principles of simple interest, because the number of years purchase is indefinite. For, as a finite quantity can make no alteration upon an indefinite one, by any operation of arithmetic, the fractional part of the annuity, (Equation 1st, 29.) viz. $1 + \frac{t-1}{2}r$ $\frac{1}{1+tr} = 1$, when t is indefinite; and of consequence, the number of years purchase of a perpetuity, viz. $t \times 1$, is also indefinite. Or thus, seeing the value of an annuity at simple interest is supposed to be equal to the present value of the amount of the several annual payments, with their interest, collected into one sum; but as this, when t is indefinite, cannot be done, therefore the value of a perpetuity upon the principles of simple interest must remain impracticable, and its only value that can be obtained, must be derived from the principles of compound interest, viz. $\frac{a}{r}$. (56.)

C H A P. IV.

Of GEOMETRICAL PROGRESSION.

35. WHEN a series of quantities, exceeding two in number, increases by the same common multiplier, or decreases by the same common divisor, the terms of the series form a continued Geometrical Progression; and the common multiplier or divisor is called the common ratio of the series. Thus,

$a, ar, ar^2, ar^3, ar^4, \dots, ar^{n-1} = z.$ an increasing series.

$Z, \frac{z}{r}, \frac{z}{r^2}, \frac{z}{r^3}, \frac{z}{r^4}, \dots, \frac{z}{r^{n-1}} = a.$ a decreasing series.

$1, 2, 4, 8, 16, 32, 64,$ 2 the common { Multiplier.
 $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64},$ { Divisor.

36. In a geometrical series, the product of the extremes is equal to the product of any two terms

terms equally distant from the extremes; thus $az = ar \times \frac{z}{r}$; and the square of any one term is equal to the rectangle under any two other terms equally distant from it; thus, $ar^2 \times ar^2 = ar \times ar^3$.

37. In a geometrical series, let

a represent the 1st term.

z the last term.

n the number of terms.

r the common ratio = quot. of the greater by the lesser contig. term.

s the sum of the series, and L , the Logarithm of any quantity.

Then, as r is always increasing from the 2d term, its exponent in the last term will be $n-1$, and $ar^{n-1} = z$.

38. In a geometrical series, the sum of all the terms wanting the least is equal to the sum of all the terms wanting the greatest multiplied by the common ratio.

that is, $s - a = \frac{s - z}{r} \times r$, in an increasing series.
and $s - a = \frac{s - z}{r} \times \frac{1}{r}$, in a decreasing series.

for $\frac{ar, ar^2, ar^3 \dots}{r \times a, ar, ar^2 \dots} = \frac{ar^{n-1}}{ar^{n-2}}$ Hence,

39. In a continued geometrical progression, if any three of the five quantities, a , z , n , r , and s be given, the other two may be determined, as in the following table:

G

TABLE:

TABLE.

Given.		
$a, r, n,$	$s = a \times \frac{r^n - 1}{r - 1}, \dots$	Equation 1.
$a, r, z,$	$s = \frac{rz - a}{r - 1} = \frac{z - a}{r - 1} + z, \dots$	2.
$a, n, z,$	$s = \frac{z - a}{n - 1 \sqrt{\frac{z}{a} - 1}} + z, \dots$	3.
$r, z, n,$	$s = \frac{z}{r^n - 1} \times \frac{r^n - 1}{r - 1}, \dots$	4.
$a, r, n,$	$z = ar^{n-1}, \dots$	5.
$a, r, s,$	$z = \frac{rs - s + a}{r} = s - \frac{s + a}{r}, \dots$	6.
$r, n, s,$	$z = r^{n-1} s \times \frac{r - 1}{r^n - 1}, \dots$	7.

Sought s in Equation 1.		Sought z in Equation 7.	
38	$s - a = s - z \times r.$	38.	$s - \frac{z}{r^{n-1}} = r s - r z,$
37	$z = ar^{n-1}.$		$rz - \frac{z}{r^{n-1}} = s \times r - 1.$
	$s - a = rs - ar^n.$		$zr^n - z = r^{n-1} s \times r - 1.$
	$s \times r - 1 = a \times r^n - 1.$		
Sought s in Equation 3.			
37	$ar^{n-1} = z.$		
	$r^{n-1} = \frac{z}{a}.$		
	$r = \frac{n-1}{n-1} \sqrt{\frac{z}{a}}.$		
	Substit. this in Equ. 4.		

Given:

Given.		
$r, z, n,$	$a = \frac{z}{r^{n-1}}, \dots$	Equation 8.
$r, z, s,$	$a = rz - s \times \overline{r-1}, \dots$	9.
$r, n, s,$	$a = s \times \frac{r-1}{r^{n-1}}, \dots$	10.
$a, r, z,$	$n = \frac{L.z}{L.r} + 1 = \frac{L.z - L.a}{L.r} + 1, \dots$	11.
$a, r, s,$	$n = \frac{L.\overline{r-1} \times s + a}{L.r} = \frac{L.\overline{r-1} \times s + a - L.a}{L.r},$	12.
$a, z, s,$	$n = \frac{L.z - L.a}{L.s - a - L.s - z} + 1, \dots$	13.
$r, z, s,$	$n = \frac{L.z - \overline{L.rz - s \times \overline{r-1}}}{L.r} + 1, \dots$	14.

Sought n in Equation 11.		Sought n in Equation 13.	
37.	$r^{n-1} = \frac{z}{a}.$	Equ. 15.	$r = \frac{z-a}{s-a}.$ Substit. this in Equ. 11.
	$n-1 = \frac{L.z}{L.r}.$		Sought n in Equation 14.
		Equ. 9.	$a = rz - s \times \overline{r-1}.$ Substit. this in Equ. 11.
Sought n in Equation 12.			
38.	$s-a = rs - ar^n.$		
	$ar^n = rs - s + a.$		
	$r^n = \frac{r-1 \times s + a}{a}.$		

Given.			
$a, z, s,$	$r = \frac{s-a}{s-z}, \dots$		Equation 15.
$a, n, s,$	$\frac{s}{a} r - r^n = \frac{s}{a} - 1, \dots$		16.
$a, n, z,$	$r = \sqrt[n-1]{\frac{z}{a}}, \dots$		17.
$n, z, s,$	$\frac{s}{s-z} r^{n-1} - r^n = \frac{z}{s-z}, \dots$		18.

In the above table, the expressions belong to an increasing progression, and may be applied also to a decreasing one, by making the quantities a and z change places every where.

EXAMPLES.

1st, In the geometrical series, 1, 2, 4, 8, 16, &c. containing sixteen terms, sought, the last term and the sum of the series. Here $a=1, r=2, n=16$. Sought, z and s .

$$\text{Equation 1. } a \times \frac{r^n - 1}{r - 1} = 1 \times \frac{65526 - 1}{1} = 65525 = s. \quad \begin{array}{l} L. r = 0.3010300 \\ n = 16 \\ \hline 4.8164800 = \\ 65526 = r^n \end{array}$$

$$\text{Equation 5. } ar^{n-1} = 1 \times 32768 = z.$$

Sought r in Equation 16.

$$\begin{array}{l} 38. \quad \left| \begin{array}{l} s - a = rs - ar^n, \\ rs - ar^n = s - a, \\ \frac{s}{a} r - r^n = \frac{s}{a} - 1. \end{array} \right. \end{array}$$

Sought r in Equation 18.

$$\begin{array}{l} 38. \quad \left| \begin{array}{l} 1 - \frac{z}{r^{n-1}} = r - zr^n, \\ zr^{n-1} - z = r - z \times r^n, \\ \frac{s}{s-z} r^{n-1} = \frac{z}{s-z} + r^n. \end{array} \right. \end{array}$$

2d, A gentleman, who had a daughter married, gave the husband towards her portion 4 s. promising to triple that sum the first day of every month, for nine months after the marriage. The sum paid on the first day of the ninth month was 26244 shillings. What was the lady's portion? Here are given a , r , z ; sought s .

$$\text{Equ. 2. } \frac{rz-a}{r-1} = \frac{78728}{2} = 39364 \text{ sh.} = \text{L. } 1968, 4 \text{ s.} = s.$$

3d, A gentleman buys a fine house, in which were 20 thresholds, and, in name of price, was to lay a farthing on the first threshold, a half-penny on the second, a penny on the third; doubling the sum on every following threshold; what would the house cost him? Here are given, a , r , n ; sought s .

$$\text{Equ. 1. } a \times \frac{r^n - 1}{r - 1} = 1048575 \text{ Far.} = \text{L. } 1092, 5 \text{ s. } 4 \text{ d.} = \text{Price.}$$

4th, A Gentleman sold his estate for L. 21844, which was discharged by several payments in geometrical progression; the first was L. 4, the last L. 16384; what was the ratio, and how many payments were there? Here are given a , z , s ; sought r , n .

$$\text{Equation 15. } \frac{s-a}{s-z} = \frac{2184}{546} = 4 = \text{Ratio.}$$

$$\text{Equation 13. } \frac{L.z - L.a}{L.s - L.a} + 1 = \frac{3.61236}{.60206} + 1 = 7 = n.$$

Of a Geometrical Series continued indefinitely.

40. As in a geometrical series $s - a = s - z \times r$, (38), hence $1 : r :: s - z : s - a$, and $1 : r - 1 :: s - z : z - a$. but as the quantities a, z , change places every where, when applied from an increasing to a decreasing series, we shall have $1 : r - 1 :: s - a : a - z$; therefore supposing z to vanish, or the decreasing series to have no last term, then $1 : r - 1 = s - a : a$; which gives $s = \frac{ra}{r-1}$; hence the following

T A B L E.

Given.		
$a, r,$	$s = \frac{ra}{r-1} = ra \times \frac{1}{r-1}, \dots$	Equation 1.
$a, s,$	$r = \frac{s}{s-a} = s \times \frac{1}{s-a}, \dots$	2.
$r, s,$	$a = \frac{s \times r - 1}{r} = s \times \frac{1}{r-1}, \dots$	3.

EXAMPLES.

1st, Required the sum of the series, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \&c.$

Here $a = 1, r = 2$, and $s = \frac{2 \times 1}{1} = 2$, the sum.

2^d, Required the sum of the series $.333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}, \&c.$

Here $a = \frac{3}{10}, r = 10$, and $s = \frac{3 \times 10}{10 \times 10 - 1} = \frac{1}{3}$, the sum.

3^d, $\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} \&c. = \frac{R \times \frac{1}{R}}{R-1} = \frac{1}{R-1} = \frac{1}{r}.$

CHAP.

C H A P. V.

Of COMPOUND INTEREST.

41. **W**HEN the interest of money at the end of each year is added to the principal sum, and both bear interest the following year, money is said to be at compound interest: In which let

p represent any principal sum lent out.

t the time.

r the interest
 R the amount } of L. 1 for one year.

s the amount of principal and interest; and L the logarithm of a quantity.

Then L. 1, at the end of the year, will become $1+r=R$, hence the geometrical series $1 : R : R^2 : R^3 : R^4 \dots R^t$. So R^t will be the amount of L. 1, for the time t , and at the rate r . But $1 : p :: R^t : pR^t = s$, the amount of p , at the end of t years, at the rate r ; hence the following table.

TABLE.

TABLE.

Given.		
$p, R, t,$	$s = pR^t$ or $L.p + L.R \times t = L.s \dots$	Equa. 1.
$s, R, t,$	$p = \frac{s}{R^t}$ or $L.s - L.R \times t = L.p, \dots$	2.
$s, R, p,$	$t = \frac{L.s - L.p}{L.R}, \dots$	3.
$s, p, t,$	$R = \sqrt[t]{\frac{s}{p}}$ or $L.R = \frac{L.s - L.p}{t}, \dots$	4.

Ex. 1. Required, the amount of L. 25 foreborn 12 years, at 5 per cent. compound interest. Here, $p = 25, t = 12, R = 1.05$.

$$\begin{array}{r} L.p = 1.3979400 \\ + L.R \times t = .2542716 \end{array}$$

$$\begin{array}{r} L.R = 0.0211893 \\ t = 12 \end{array}$$

$$\text{Sum} = 1.6522116 = L. 44.896 \quad .2542716$$

2d. Sought, the principal which, in the space of 12 years, will amount to L. 300, reckoning interest at 4 per cent. Here,

$$\begin{array}{r} L.s = 2.4771213 \\ - L.R \times t = .2044002 \end{array}$$

$$\begin{array}{r} L. 1.04 = 0.0170333 \\ t = 12 \end{array}$$

$$\text{Diff.} = 2.2727211 = L. 187.379 \quad .2044002$$

3d. Sought, the time in which L. 15.875 will amount to L. 31.9436 at the rate of 6 per cent.

$$\frac{L.s - L.p}{L.R} = \frac{.3036708}{.0253059} = 12 \text{ years.}$$

4th. At what rate of interest per cent. will L. 480 amount to L. 643.246, in the space of 6 years.

$$\frac{L.s - L.p}{t} = \frac{.1271358}{6} = .0211893 = L. 1.05. = 5 \text{ per cent.}$$

5th.

5th. Required the amount of one farthing lent out at 5 per cent. compound interest for the space of 1000 years? Here,

$$\begin{array}{r} L. 1.05 \times 1000 = 21.1893 \\ + L. .00104168 = -3.0177287 \end{array}$$

$L. pR^t = 18.2070287 = L. 1610752, \&c. \text{ to } 19 \text{ places of figures.}$

If p the principal consists of pounds, shillings, and pence, or t the time, of years, months, and days, these parts of an integer may be thrown in-
to

OF LOGARITHMS.

Logarithms are a most valuable invention for solving questions in trigonometry, or where the higher powers of numbers, as here, are concerned; and for which the learned world are indebted to John Napier, Baron of Merchiston, in Scotland, about the year 1614.

1st. The index of the logarithm of any number is always one less than the number of integers it contains; of consequence the index of the logarithm of any one of the 9 digits is 0; and the index of the logarithm of a decimal fraction is negative, though still equal to the distance of its first significant figure from the units place, being -1, -2, -3, &c. or the complement thereof to 10, viz. 9, 8, 7, &c.

2d. The logarithm of any number to 7 places of figures may be found by Sherwin's Tables, namely of 5 figures by the Tables, and of two by the column *pts*, nearly exact, and *vice versa*. Thus,

$$\begin{array}{r} L. 6484.300 = 3.8118631 \\ L. .080 = 53 \\ L. .006 = 4 \end{array}$$

$$L. 6484.386 = 3.8118688$$

3d. The multiplication or division of natural numbers is performed by the addition or subtraction of their logarithms, or, in place of subtraction, the addition of the complement of a logarithm may be used, providing you cast 10 out of the index of the sum. Thus,

$$\begin{array}{r} L. 20 = 1.3010300 \\ L. 21 = 1.3222193 \\ \text{Co. } L. 220 = 7.6575773 \end{array}$$

$$L. \frac{20 \times 21}{220} = 0.2808266 = 1.909$$

H

4th. A.

to a decimal fraction, and the answer found as above.

Ex. Sought the amount of L. 310:12:6, for 5 years and 91 days, at 4 per cent. compound interest. Here,

$$pr = L. 1.04 \times 5.25 + L. 310.625 = L. 381, 12 s. 11 d. = \text{Amount.}$$

42. The several particulars respecting compound interest may more readily be found by the help of the three following Tables, wherein are calculated the amount and present worth of L. 1 for any given time, and at several rates of compound interest.

4th. A given power of any number may be found by multiplying the logarithm of the number by the index of the power, the product will be the logarithm of the power sought. Thus,

$$L. 34^3 = 1.50515 \times 3 = 4.5154500 = 32763, \text{ the cube of } 32.$$

In a decimal fraction, the first significant figure of the power must be put so many places below the place of units, as the index of its logarithm wants of 10 $\times$ index of the power. Thus,

$$L. .25^4 = 9.39794 \times 4 = 37.5917600 = .00390625, \text{ seeing index } \times 10 = 40.$$

5th. The root of any power may be extracted by dividing the logarithm of the number by the index of the power, the quotient gives the logarithm of the root sought. Thus,

$$L. \sqrt[3]{2304} = \frac{3.3624823}{2} = 1.6812412 = 48, \text{ the root sought.}$$

In a decimal fraction, prefix to the index of its logarithm a figure less by one than the index of the power, and divide as above. Thus,

$$L. \sqrt[3]{.75} = \frac{20.8710613}{3} = 9.9583533 = .90856, \text{ the root sought.}$$

6th. Any number of mean proportionals, between any two given numbers, may be found thus: Let $a = 1st$, $z = last$ number, $n =$ number of proportions; then $\frac{L. z - L. a}{n-1} = L. r$, which added to $L. a$, n times, gives the proportionals.

TABLE

TABLE I. Shewing the Amount of L. 1 for Days.

Dys	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1	1.0000809	1.0000942	1.0001074	1.0001206	1.0001336	1.0001596
2	1.0001619	1.0001880	1.0002149	1.0002412	1.0002673	1.0003193
3	1.0002429	1.0002827	1.0003224	1.0003618	1.0004011	1.0004790
4	1.0003240	1.0003770	1.0004299	1.0004822	1.0005348	1.0006387
5	1.0004050	1.0004713	1.0005374	1.0006033	1.0006685	1.0007985
6	1.000486	1.000565	1.000643	1.000724	1.000802	1.000958
7	1.000567	1.000660	1.000752	1.000844	1.000936	1.001118
8	1.000648	1.000754	1.000860	1.000965	1.001070	1.001278
9	1.000729	1.000848	1.000967	1.001086	1.001203	1.001437
10	1.000810	1.000943	1.001075	1.001206	1.001337	1.001597
20	1.001621	1.001886	1.002151	1.002415	1.002677	1.003198
30	1.002432	1.002831	1.003228	1.003624	1.004018	1.004801
40	1.003244	1.003777	1.004307	1.004835	1.005361	1.006806
50	1.004057	1.004723	1.005387	1.006048	1.006705	1.008014
60	1.004871	1.005671	1.006468	1.007262	1.008052	1.009624
70	1.005685	1.006619	1.007550	1.008477	1.009401	1.011237
80	1.006499	1.007568	1.008633	1.009694	1.010751	1.012853
90	1.007315	1.008518	1.009717	1.010912	1.012103	1.014471
100	1.008131	1.009469	1.010803	1.012132	1.013457	1.016092
120	1.009765	1.011374	1.012978	1.014576	1.016169	1.019341
140	1.011402	1.013282	1.015157	1.017026	1.018890	1.022601
160	1.013041	1.015194	1.017341	1.019482	1.021617	1.025871
180	1.014683	1.017109	1.019530	1.021944	1.024352	1.029152
200	1.016328	1.019028	1.021723	1.024412	1.027095	1.032443
220	1.017976	1.020951	1.023921	1.026885	1.029844	1.035745
240	1.019626	1.022877	1.026124	1.029365	1.032601	1.039057
260	1.021278	1.024807	1.028332	1.031851	1.035365	1.042380
280	1.022934	1.026741	1.030544	1.034343	1.038137	1.045713
300	1.024592	1.028638	1.032761	1.036840	1.040916	1.049057
320	1.026253	1.030619	1.034983	1.039344	1.043703	1.052412
340	1.027916	1.032964	1.037210	1.041854	1.046497	1.055778
360	1.029583	1.034512	1.039441	1.044370	1.049298	1.059154
365	1.03	1.035	1.04	1.045	1.05	1.06

TABLE II. Shewing the Amount of L. 1 for Years.

Yrs.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1	1.03	1.035	1.04	1.045	1.05	1.06
2	1.0609	1.071225	1.0816	1.092025	1.1025	1.1236
3	1.092727	1.108718	1.124864	1.141166	1.157625	1.191016
4	1.125509	1.147323	1.169858	1.192518	1.215506	1.262476
5	1.159274	1.187686	1.216053	1.246182	1.276281	1.338225
6	1.194052	1.22925	1.26532	1.30226	1.34009	1.41852
7	1.229874	1.27228	1.31592	1.36586	1.40710	1.50363
8	1.266770	1.31681	1.36857	1.42210	1.47745	1.59385
9	1.304773	1.36289	1.42321	1.48609	1.55133	1.68948
10	1.343916	1.41059	1.48024	1.55296	1.62889	1.79084
11	1.38423	1.45997	1.53945	1.62285	1.71034	1.89830
12	1.42576	1.51107	1.60103	1.69588	1.79585	2.01220
13	1.46853	1.56395	1.66507	1.77219	1.88565	2.13293
14	1.51259	1.61869	1.73167	1.85194	1.97993	2.26090
15	1.55796	1.67535	1.80094	1.93528	2.07893	2.39656
16	1.60471	1.73398	1.87298	2.02237	2.18287	2.54025
17	1.65284	1.79467	1.94790	2.11337	2.29202	2.69277
18	1.70243	1.85749	2.02581	2.20848	2.40662	2.85434
19	1.75350	1.92250	2.10685	2.30786	2.52695	3.02560
20	1.80611	1.98978	2.19112	2.41171	2.65329	3.20713
21	1.86029	2.05943	2.27877	2.52024	2.78596	3.39956
22	1.91610	2.13151	2.36992	2.63365	2.92526	3.60353
23	1.97358	2.20611	2.46471	2.75216	3.07152	3.81975
24	2.03279	2.28333	2.56330	2.87601	3.22510	4.04893
25	2.09378	2.36324	2.66583	3.00543	3.38635	4.29187
26	2.15659	2.44595	2.77247	3.14068	3.55567	4.54938
27	2.22129	2.531567	2.88337	3.28201	3.73345	4.82234
28	2.28792	2.62017	2.99870	3.42969	3.92013	5.11168
29	2.35656	2.711878	3.11865	3.58403	4.11613	5.41838
30	2.42726	2.80679	3.24339	3.74531	4.32194	5.74349
31	2.50008	2.905031	3.37313	3.91385	4.53804	6.08810
32	2.57508	3.00671	3.50806	4.08998	4.76494	6.45338
33	2.65233	3.11194	3.64838	4.27403	5.00319	6.84059
34	2.73190	3.22086	3.79431	4.46636	5.25335	7.25102
35	2.81386	3.33359	3.94609	4.66734	5.51601	7.68608
36	2.89828	3.45026	4.10393	4.87738	5.79181	8.14725
37	2.98522	3.57102	4.26809	5.09686	6.08141	8.63608
38	3.07478	3.69601	4.43881	5.32622	6.38547	9.15425
39	3.16703	3.82537	4.61636	5.56589	6.70475	9.70351
40	3.26204	3.95926	4.80102	5.81636	7.03998	10.28571
41	3.35989	4.09783	4.99306	6.07810	7.39198	10.90286
42	3.46069	4.24126	5.19278	6.35161	7.76158	11.55703
43	3.56451	4.38970	5.40050	6.63744	8.14966	12.25045
44	3.67145	4.54334	5.61651	6.93612	8.55715	12.98548
45	3.78159	4.70236	5.84117	7.24825	8.98500	13.76461
46	3.89504	4.86694	6.07482	7.57442	9.43226	14.59048
47	4.01189	5.03728	6.31781	7.91527	9.90597	15.46591
48	4.13225	5.21356	6.57053	8.27145	10.40127	16.39387
49	4.25622	5.39606	6.83335	8.64367	10.92133	17.37750
50	4.38390	5.58494	7.10668	9.03263	11.46740	18.42015

Table I. and II. shew the amount of L. 1 for days or years, and are constructed by Equation 1st, 41. whereby the tabular number $n = R^t$. Or thus, Table I. is calculated by taking 364 mean geometrical proportionals, betwixt one, and the amount of L. 1 for one year, according to the several rates of interest, or, which is the same thing, by dividing the logarithm of R by 365, the natural number answering to the quotient will be the amount of L. 1 for one day; and this amount multiplied into itself continually will give the amount for any number of days. If a given number of days be not in the Table, divide it into two such numbers as are to be found there, multiply their tabular numbers into one another, and their product will be the amount of L. 1 for the given time. In like manner, Table II. is constructed by multiplying the amount of L. 1 for one year, at a given rate, by itself continually, the products of which shew the amount of L. 1 for their corresponding times. And if the time consists of years and days, their tabular numbers must be multiplied into one another to form the amount of their sum. From the nature of the Tables, $1 : n :: p : np = s$, the amount of any principal p , for the time and at the rate corresponding to n . Hence the following

TABLE.

Given,	
$p, t, r,$	$s = np.$
$s, t, r,$	$p = \frac{s}{n}.$
$t, r, p, \}$	$n = \frac{s}{p}.$
$s, t, p, \}$	

r being given, t is found opposite to n ; and t being given, r is found above n in the tables.

Ex. 1st,

Ex. 1st. What will L. 500 amount to in 15 years, at five per cent. compound interest? Here,

$$np = 2.07893 \times 500 = \text{L. } 1039.465.$$

Ex. 2d. What will be the amount of L. 419.2965 in the space of 20 years and 9 months, at $4\frac{1}{2}$ per cent. Here,

$$n \times n = 2.49265 = 0.3966608$$

$$p = 419.2965 = 2.6224885$$

$$\text{L. } np = 3.0191493 = \text{L. } 1045.08.$$

Ex. 3d. What principal will amount to L. 1039.464 in 15 years, at 5 per cent? Here,

$$\frac{s}{n} = \frac{1039.464}{2.07893} = \text{L. } 500.$$

Ex. 4th. At what rate of interest will L. 25 amount to L. 44.896 in the space 12 years? Here,

$$\frac{s}{p} = \frac{44.896}{25} = 1.79585, \text{ found under 5 per cent.}$$

Ex. 5th. In what time will L. 300 amount to L. 500, at the rate of 5 per cent? Here,

$$\frac{s}{p} = \frac{500}{300} = 1.66666, \text{ corresponding to 10 years.}$$

$$\text{and } \frac{1.66666}{1.62889} = 1.0232, \text{ corresponding to 172 days.}$$

TABLE

TABLE III. Shewing the present Worth of
L. 1 for Years.

Yrs.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1	.970874	.966183	.961538	.956938	.952381	.943396
2	.942596	.933510	.924556	.915729	.907029	.889996
3	.915141	.901942	.888996	.876296	.863837	.839619
4	.888487	.871442	.854804	.838561	.822702	.792093
5	.862608	.841973	.821927	.802451	.783526	.747258
6	.837484	.813500	.790314	.767895	.746215	.704960
7	.813091	.785991	.759918	.734828	.710681	.665057
8	.789409	.759411	.730690	.703185	.676839	.627412
9	.766416	.733731	.702586	.672904	.644609	.591898
10	.744093	.708918	.675564	.643927	.613913	.558394
11	.722421	.684945	.649581	.616198	.584679	.526787
12	.701379	.661783	.624597	.589663	.556837	.496969
13	.680950	.639404	.600574	.564271	.530321	.468839
14	.661117	.617781	.577475	.539973	.505068	.442301
15	.641860	.596190	.555264	.516720	.481017	.417265
16	.623167	.576706	.533908	.494469	.458111	.393646
17	.605016	.557203	.513373	.473176	.436296	.371364
18	.587394	.538361	.493628	.452800	.415520	.350344
19	.570286	.520155	.474642	.433302	.395734	.330513
20	.553675	.502566	.456387	.413643	.376889	.311804
21	.537549	.485571	.438833	.396787	.358942	.294155
22	.521892	.469150	.421955	.379701	.341849	.277503
23	.506691	.453285	.405726	.363350	.325571	.261797
24	.491933	.437957	.390121	.347703	.310068	.246978
25	.477605	.423147	.375117	.332730	.295303	.232998
26	.463694	.408837	.360689	.318402	.281240	.219810
27	.450189	.395012	.346816	.304691	.267848	.207368
28	.437076	.381654	.333477	.291571	.255093	.193630
29	.424346	.368748	.320651	.279015	.242946	.181556
30	.411986	.356278	.308318	.267000	.231377	.171110
31	.399987	.344230	.296460	.255502	.220359	.164255
32	.388337	.332589	.285058	.244500	.209866	.154457
33	.377026	.321342	.274094	.233971	.199872	.146186
34	.366045	.310476	.263552	.223896	.190355	.137911
35	.355383	.299977	.253415	.214254	.181290	.130105
36	.345032	.289833	.243668	.205028	.172657	.122741
37	.334983	.280031	.234297	.196199	.164435	.115793
38	.325226	.270562	.225285	.187750	.156605	.109238
39	.315753	.261412	.216620	.179665	.149148	.103055
40	.306556	.252572	.208289	.171928	.142045	.097222
41	.297628	.244031	.200278	.164525	.135281	.091720
42	.288959	.235779	.192575	.157440	.128840	.086527
43	.280543	.227806	.185168	.150660	.122704	.081629
44	.272372	.220102	.178046	.144173	.116861	.077009
45	.264438	.212660	.171198	.137964	.111296	.072650
46	.256736	.205468	.164614	.132023	.105996	.068538
47	.249258	.198520	.158282	.126338	.100949	.064658
48	.241998	.191806	.152195	.120897	.096142	.060998
49	.234950	.185320	.146341	.115691	.091564	.057545
50	.228107	.179053	.140711	.110709	.087204	.054288

Table III shews the present value of L. 1 for years, and is constructed by Equation 2d, 41. whereby the tabular number $n = \frac{1}{R^t}$. Or it is the reciprocal of Table II. whereby n of Table III. = $\frac{1}{n}$ of Table II. From the nature of the Table $1 : n :: s : ns = p$, the principal or present worth of s , for the time and at the rate corresponding to n . Hence the following

TABLE.

Given.	
$s, t, r,$	$p = ns.$
$p, t, r,$	$s = \frac{p}{n}.$
$s, r, p,$	$n = \frac{p}{s}.$
$s, t, p,$	

Ex. 1. Required the present worth of L. 250, due 10 years hence, discounting at the rate of $4\frac{1}{2}$ per cent. compound interest.

$ns = .643927 \times 250 = \text{L. } 160.98175$ present worth.

Ex. 2. In what time will L. 350 amount to L. 571.458, at 4 per cent compound interest?

Here $\frac{p}{s} = \frac{350}{571.458} = .61247$ corresponding to 12 years and an half.

These examples may, with equal facility, be determined by the equations annexed to Table II.

Remarks upon Compound Interest.

43. 1st. A sum of money put out at compound interest ought, by the doctrine of fluxions, to accumulate every instant, at the rate of the instant; so as to resemble an organized body in its gradual and constant increase. Upon this supposition let $\dot{r}$ be the fluxion of the rate, or the rate of the instant,

Then $1 + \dot{r}$ = amount of L. 1 in one instant multiplied by $1 + \dot{r}$

produces $1 + 2\dot{r} + \dot{r}\dot{r}$ = the fluxion of that amount at the end of the first year; whose fluent, viz. $1 + 2r$ = amount of L. 1 at the end of the year. Hence the interest of money, when the times and rates are the same, will, when it accumulates every instant, be double to what it would be when it accumulates only once in the year. But as this mode might elude the grasp of calculation, and for the benefit of all concerned, the accumulation of money hath been extended to the twelvemonth, and only takes place once in the year.

2d. Although there be no particular theorem for discovering the compound interest of a sum of money distinct from the amount, yet it may easily be found, being the difference betwixt the principal and the amount. In like manner, was it lawful to discount money on the principles of compound interest, the discount might be found by subtracting the present worth from the sum of the bill.

3d. If the time be sought in which a sum of money will double itself at the several rates of compound interest; in this case, as the amount is equal to twice the principal, $pR^t = 2p$, which gives $R^t = 2$, and $t = \frac{L \cdot 2}{L \cdot R}$. Hence,

$$s = 2p, \text{ when } t = \frac{L \cdot 2}{L \cdot R} = \begin{array}{r|l} 23.45- & 3 \\ 20.149- & 3\frac{1}{2} \\ 17.67+ & 4 \\ 15.8- \text{ years at } & 4\frac{1}{2} \text{ per cent.} \\ 14.206+ & 5 \\ 11.98+ & 6 \\ 34.886+ & 2 \end{array}$$

4th. The amount of a sum of money, *ceteris paribus*, is always greater, and its present value less, upon the principles of compound than of simple interest, in proportion of $\frac{1}{1+r^t}$, to $\frac{1}{1+tr}$; and of $\frac{1}{1+r^t}$, to $\frac{1}{1+tr}$. And as this difference is very considerable, seeing $\frac{L \cdot 2}{L \cdot R}$ years interest of the one is equal to $\frac{1}{r}$ years interest of the other, the charging compound interest upon any bill hath been expressly prohibited by acts of parliament, at the risque of forfeiting both principal and interest. Notwithstanding of this, compound interest enters deeply into the transactions of men in their affairs of property. Thus, not only the sale of annuities, and the purchase of perpetuities are calculated upon the principles of compound interest; but also where cash-accounts are balanced, or the interest of money and the rent of an estate received yearly, there you have the advantage of compound interest, seeing you have the use, and can dispose of that interest.

C H A P. VI.

Of ANNUITIES computed at COMPOUND INTEREST.

1st, Of the Amount of an Annuity in Arrears.

44. IN computing annuities at compound interest, let a represent an annuity, pension, or rent, &c. in arrears; then, since the first year's annuity bears no interest, the last year's annuity will be represented by aR^{t-1} ; and deriving a series from the quantity a , in the ratio of $1 : R$, we shall have $a + aR + aR^2 + aR^3 + aR^4 \dots + aR^{t-1} = s$ which gives $s = a \times \frac{R^t - 1}{R - 1}$. Hence the following

T A B L E.

Given.		
$a, t, R,$	$s = a \times \frac{R^t - 1}{R - 1} = a \times \frac{R^t - 1}{r} = \frac{a}{r} R^t - \frac{a}{r}$	Equat. 1.
$s, t, R,$	$a = rs \times \frac{1}{R^t - 1} \dots$	2.
$s, t, a,$	$\frac{s}{a} R - R^t = \frac{s}{a} - 1 \dots$	3.
$s, R, a,$	$t = \frac{L. \frac{rs}{a} + 1}{L. R} \dots$	4.

Sought s in Equation 1st.

$$\begin{aligned}
 38. \quad & s - a = \frac{1 - aR^{t-1}}{1 - R} \times R. \\
 & s - a = R s - aR^t. \\
 & R s - s = aR^t - a. \\
 & \overline{R - 1} \times s = a \times \frac{R^t - 1}{R - 1}. \\
 & \text{but } R = 1 + r, \text{ \&c.}
 \end{aligned}$$

Eq. 2.

Sought R in Equation 3.

$$\begin{aligned}
 38. \quad & \overline{R^t - 1} \times a = R s - s. \\
 & R^t - 1 = \frac{s}{a} R - \frac{s}{a}.
 \end{aligned}$$

Sought t in Equation 4th.

$$\overline{R^t - 1} \times a = rs.$$

$$R^t - 1 = \frac{rs}{a} \text{ \& } R^t = \frac{rs}{a} + 1.$$

Ex. 1st. It is required to find the amount of an annuity L. 5.25 per annum, at the end of 30 years, reckoning interest at 4 per cent? Here $a = 5.25$, $t = 30$, $R = 1.04$ and $r = .04$.

$$a \times \frac{R^t - 1}{r} = 5.25 \times \frac{2.243}{.04} = \begin{array}{r} L. R = .0170333 \\ t = 30 \end{array}$$

$$= L. 297.42 \text{ the amount.} \quad L. R^t = .5109990 = \begin{array}{r} 3.2433 \\ 1 \end{array}$$

$$R^t - 1 = \begin{array}{r} 2.2433 \end{array}$$

Ex. 2d. Sought, the yearly annuity which in the space of 15 years will amount to L. 1000, at the rate of $4\frac{1}{2}$ per cent. Here, by Equation 2d,

$$rs \times \frac{1}{R^t - 1} = 45 \times \frac{1}{.93528} = \begin{array}{r} L. R = .0191163 \\ t = 15 \end{array}$$

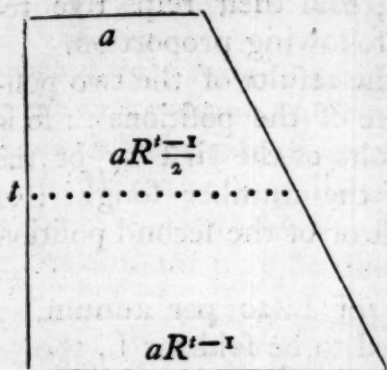
$$45 \times 1.0692 = L. 48.114 \quad \begin{array}{r} .2867445 = 1.93528 \\ 10 \end{array}$$

$$L. .93528 = 9.9709416 \quad \begin{array}{r} .93528 = R^t - 1 \\ .0290584 = 1.0692 = \frac{1}{R^t - 1} \end{array}$$

Ex. 3d. In what time will an yearly annuity of L. 48.114 amount to L. 1000 at the rate of $4\frac{1}{2}$ per cent. compound interest?

$$\text{Equation 4th, } \frac{L. \frac{rs}{a} + 1}{L. R} = \frac{L. 1.93528}{L. 1.045} = \frac{.2867445}{.0191163} = 15 \text{ years.}$$

45. Given, the annuity, time, and amount; sought, the rate of compound interest. In this case, as R , in Equation 3d, is involved to a high power, and the Equation is of a mixed nature, the



the value of R may be found in this manner. The amount of an annuity may be conceived to be nearly equal to the area of a geometrical figure, having two of its sides parallel and two of its angles right; whose least breadth =

a , its greatest = aR^{t-1} , and its height = t . Hence $s = atR^{\frac{t-1}{2}}$, nearly; which gives $\frac{s}{at} = R^{\frac{t-1}{2}}$, and $\frac{t-1}{2} \sqrt{\frac{s}{at}} = R = 1 + r$, nearly.

Ex. 4th. An annuity of L. 34.4 in arrears 12.5 years, amounts to L. 614.4328; it is required to find the rate of interest allowed. Here

$$L. \frac{t-1}{2} \sqrt{\frac{s}{at}} = .0269575 = L. s = 2.7884743$$

$$1.064 = R = 1 + r = L. at = 2.6334685$$

6 per cent. nearly. $\frac{t-1}{2} = 5.75) 0.1550058 (.0269575$

If greater exactness be desired, let $\frac{6}{1+r} = b$, then $\sqrt{bb + 2rb} - b = r$ corrected. Or thus *,

Assume

* *The Rule of DOUBLE POSITION.*

Let P and p , the two positions, and x the number sought, be equally increased or diminished, by any operation of arithmetic, producing thereby M , m and $\mathcal{Q}$, their respective results; then it is obvious, that

$$M \oslash m : P \oslash p :: \begin{cases} M \oslash \mathcal{Q} : P \oslash x. \\ m \oslash \mathcal{Q} : p \oslash x. \end{cases}$$

Let

Assume two positions, find their respective results, and work by the following proportion.

As the difference of the results of the two positions : is to the difference of the positions :: so is the difference of the results of the first or of the second position, and of the number sought : to the difference of the first or of the second position and number sought.

Ex. 5th. An annuity of L. 20 per annum, 7 years in arrears, is offered to be sold for L. 180.5; what rate of interest is allowed to the purchaser in this bargain? Here R is sought.

1st position, $R = 1.08 \dots$ result 178.456 $\dots$ error $- 2.044$

2d position, $R = 1.085 \dots$ result 181.210 $\dots$ error $+ 0.710$

By the question

result 180.500, diff. of res. = 2.754

hence

Let $M \propto Q = R$, and $m \propto Q = r$ be the errors of the two results M and m , which are either $+$ or $-$; then

$$M \propto m = \left\{ \begin{array}{l} R \propto r \\ R + r \end{array} \right\} \text{ when the errors are } \left\{ \begin{array}{l} \text{alike.} \\ \text{unlike.} \end{array} \right.$$

Ex. Suppose $2x + \frac{x}{2} + 2 + \frac{1}{3} = 20$; sought x .

1st position, $P = 9 \dots M = 25 \dots R = + 5.0$

2d position, $p = 6 \dots m = 17.5 \dots r = - 2.5$

sought $x \dots Q = 20 \dots M \propto m = 7.5$

hence, $7.5 : 3 :: 5 : 2 = P \propto x$, which gives $x = 7$, when the errors $+$ and $-$ are equal, $\frac{P+p}{2} = x$.

When the higher powers of the unknown quantity, x , are concerned in the question, this method can only give an answer near the truth; seeing $P^2 : p^2$ is in a duplicate, and $P^3 : p^3$, in a triplicate ratio of $P : p$.

The above proportion is also the foundation of a second well-known rule in double position.

R U L E

hence $2.754 : .005 :: 2.044 : .0037$, to be added to the first position, making $R = 1.0837$; which shews that the rate of interest is L. 8.37 per cent.

46. When an annuity is paid, which is usually the case, every half-year, quarterly, or monthly, its amount may be found by the Equation, $s = a \times \frac{R^t - 1}{R - 1}$; Here a is the half-year's, or quarterly payment, R^t is the same as if the annuity had

R U L E.

Divide the difference of the products of the two positions, multiplied into their alternate errors, by the difference of their results, that is, by the difference of the errors, when they are alike; or divide the sum of the products by the difference of their results, that is, by the sum of the errors, when they are unlike; the quotient in either case will give the answer. Thus,

$$\frac{rP \propto R\rho}{M \propto m = R \propto r} = x \quad \left\{ \begin{array}{l} \text{when the errors are alike, that is both } +, \\ \text{or both } -. \end{array} \right.$$

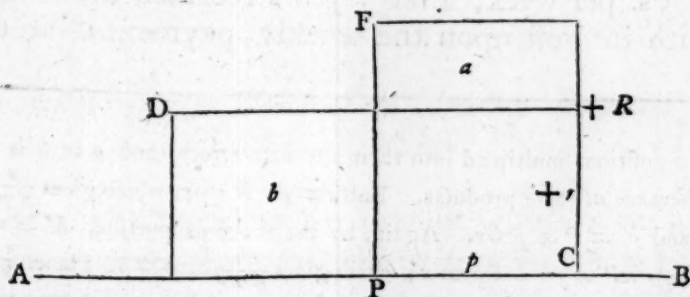
$$\frac{rP + R\rho}{M \propto m = R + r} = x \quad \left\{ \begin{array}{l} \text{when the errors are unlike, that is one } + \text{ and} \\ \text{the other } -. \end{array} \right.$$

Ex. When the errors are both $+$, or both $-$.

1st position, $= P$, its error $+ R$.

2d position, $= p$, its error $= r$.

x is fought.



Let the right line AB be conceived to terminate the true result by the question; then the errors + will fall above, and - below that line; the rectangles CD + CF = rP + Rp, are the products of the two

had been paid yearly, seeing R' for years is equal to R'' for quarters; but the divisor must always correspond to the payments, and may be discovered by the following

TABLE. Shewing the Amount of L. 1 for the even Parts of a Year.

M.	W.	D.	3 per cent.	3½ pr cent.	4 per cent.	4½ pr cent.	5 per cent.	6 per cent.
		1	1.00008098	1.0000942	1.0001074	1.0001206	1.0001336	1.0001596
		1 7	1.0005670	1.000660	1.0007524	1.0008445	1.0009361	1.001118
		2 14	1.001134	1.001320	1.001505	1.001690	1.001873	1.002138
		3 21	1.001702	1.001981	1.002258	1.002535	1.002811	1.003360
		4 28	1.002270	1.002642	1.003012	1.003382	1.003750	1.004480
1			1.002466	1.002871	1.003274	1.003675	1.004074	1.004867
3	13		1.007417	1.008637	1.009853	1.011065	1.012272	1.014674
6	26		1.014889	1.017350	1.019840	1.022253	1.024694	1.029563
9	39		1.022417	1.026137	1.029852	1.033564	1.037270	1.044671
12	52	365	1.03	1.035	1.04	1.045	1.05	1.06

In this Table, which is the same in construction with Table I. whilst the tabular numbers shew R the amount, the decimals point out r the interest of L. 1 for their corresponding days, months, or quarters, at the several rates of interest.

Ex. 1st, A servant, who was engaged to serve at 5 s. per week, after 7 years received his wages, with interest upon the weekly payments, at the

two positions multiplied into their alternate errors, and $a \oslash b$ is the difference of these products. But $a = p \times R \oslash r$, which gives $\frac{a}{R \oslash r} = p$ and $b = P \oslash p \times r$. Again, by the above proportion, $M \oslash m = R \oslash r : P \oslash p :: r : p \oslash x$; which gives $\frac{b}{R \oslash r} = p \oslash x$. Hence $\frac{a \oslash b}{R \oslash r} = (p \oslash p \oslash x) = x$; that is, the difference of the products, when the errors are alike, divided by the difference of the errors, will quot the answer. In like manner it might be shewn, when the errors are unlike, that $\frac{a+b}{R+r} = (\frac{P+p}{2} - \frac{P+p}{2} \oslash x) = x$.

rate

rate of 5 per cent. per annum ; sought, the amount of his wages, and interest thereon. Here $a = .25$, $R' = 1.4071$ and $r = .000936$. Hence,

$$a \times \frac{R'-1}{r} = .25 \times \frac{.4071}{.000936} = \text{L. } 108.734.$$

Ex. 2d. A sailor, being engaged for a voyage of 6 years, at the rate of L. 1, 10 s. per month, at his return to port, received his wages, with interest upon his monthly payments, at the rate of 4 per cent. per annum : Sought, the amount of the whole.

Here $a = 1.5$, $R = \overset{6}{1.04} = 1.26532$, and $r = .003274$.

Hence $a \times \frac{R-1}{r} = 1.5 \times \frac{.26532}{.003274} = \text{L. } 121.557.$

Ex. 3d. What will an annuity of L. 17.5, payable every quarter of a year, amount to in 5 years, at five per cent per annum ? Here,

$a = 17.5$, $R' = \overset{5}{1.05} = 1.27628$, and $r = .012272$.

Hence $17.5 \times \frac{.27628}{.012272} = \text{L. } 393.977.$

In like mannner, *mutatis mutandis*, may a , t , and R be found from the other equations of 44th, when the payments are made every half year, quarterly, &c.

2d. Of the present Worth of an Annuity computed at Compound Interest.

47. Let p represent the present worth of an annuity a ; then, since $1 : \frac{1}{R}$ the present worth of L. 1, as $a : \frac{a}{R}$, this at the end of one year will be the present worth of a ; and deriving a series from

K

the

the quantity $\frac{a}{R}$ in the ratio of $1 : \frac{1}{R}$, we shall have $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4} \dots + \frac{a}{R^t} = p$ the present worth of the annuity a , which gives $p = a \times \frac{1 - \frac{1}{R^t}}{R - 1} = a \times \frac{1 - \frac{1}{R^t}}{r}$. Or thus, the feller, when the annuity ceases, has pR^t ; the purchaser, at the end of the time, has $a \times \frac{R^t - 1}{r}$, but by supposition $pR^t = a \times \frac{R^t - 1}{r}$, which gives $p = a \times \frac{R^t - 1}{r R^t} = a \times \frac{1 - \frac{1}{R^t}}{r}$. Hence the following

T A B L E.

Given.		
$a, t, R,$	$p = a \times \frac{1 - \frac{1}{R^t}}{r} = \frac{a}{r} - \frac{a}{r R^t}$	Equation 1.
$p, t, R,$	$a = rp \times \frac{1}{1 - \frac{1}{R^t}}$	2.
$p, t, a,$	$1 + \frac{a}{p} \times R^t - R^t + 1 = \frac{a}{p}$	3.
$p, r, a,$	$t = \frac{L. a - L. a - rp}{L. R}$	4.

Sought p in Equation 1st.

$$38 \quad p - \frac{a}{R^t} = p - \frac{a}{R} \times R,$$

$$p - \frac{a}{R^t} = Rp - a.$$

$$Rp - p = a - \frac{a}{R^t}.$$

$$\overline{R-1} \times p = a \times 1 - \frac{1}{R^t}. \quad \text{Eq. 2.}$$

Sought R in Equation 3d.

$$38 \quad p - \frac{a}{R^t} = Rp - a.$$

$$Rp + \frac{a}{R^t} = a + p.$$

$$p \times R^t + 1 + a = aR^t + pR^t.$$

$$R^t + 1 + \frac{a}{p} = \frac{a}{p} R^t + R^t.$$

Sought t in Equation 4th,

$$rp = a - a \frac{1}{R^t}.$$

$$a \frac{1}{R^t} = a - rp.$$

$$a = a - rp \times R^t, \text{ and } t = \&c.$$

Ex. 1st.

Ex. 1st. What is the present worth of an yearly annuity of L. 20 to continue 20.75 years, reckoning interest at $4\frac{1}{2}$ per cent?

$$a \times \frac{1 - \frac{1}{R^t}}{r} = 20 \times \frac{.598822}{.045} = L. R^t = .3966632 \quad 1.00$$

$$L. \frac{1}{R^t} = 9.6033368 = 0.401178$$

$$L. 266.14 \text{ the present value. } 1 - \frac{1}{R^t} = .598822$$

If the years purchase which an annuity is worth, be sought; they are manifestly equal to the number of times the annuity is contained in its present value. So that n , the number of years

purchase, $= \frac{p}{u} = \frac{1 - \frac{1}{R}}{r} = n$ of Table V. Thus in the last example, $\frac{266.14}{20} = \frac{.59882}{.045} = 13.307 = \text{years purchase.}$

Ex. 2d. A gentleman is willing to spend L. 1000 upon the purchase of an annuity, to continue 15 years, being allowed discount at the rate of $4\frac{1}{2}$ per cent.; sought, the annuity.

$$rp \times \frac{1}{1 - \frac{1}{R^t}} = 45 \times 2.096 = L. R^t = .2867445 \quad 1.00$$

$$L. \frac{1}{R^t} = 9.7132555 = 0.51672$$

$$= L. 93.114 \text{ annuity. } 1 - \frac{1}{R^t} = .48328 = 9.6841988$$

$$2.0962 = .3158012$$

Ex. 3d. Sought the duration of an yearly annuity of L. 80, whose present value is L. 800; discounting interest at 5 per cent.:

$$\text{Here } \frac{L. a - L. a - rp}{L. R} = \frac{.3010300}{.0211893} = 14 \text{ years } 75 \text{ days.}$$

48. Given, the annuity, its present worth and time; sought, the rate of interest per cent.

Case 1st, If the number of years be great, as 40 or upwards, and if the rate of interest be high; in this case, $\frac{1}{R^t}$ will be of small value, and $\frac{a}{p} = r$,

with this rate find the present value of the reversion of the annuity, viz. $\frac{a}{rR^t} = x$; then $\frac{a}{p+x} = r$, the rate sought.

Ex. 4. Suppose $a = \text{L. } 70$, $p = \text{L. } 1321.3028$, and $t = 59$ years; sought r , the rate of interest.

$\frac{a}{p} = .052978 = r$; $\frac{a}{rR^t} = \frac{70}{.11139} = \text{L. } 62.844 = x$, the reversion; and $\frac{a}{p+x} = .05057 = r$ corrected = 5 per cent.

Case 2. If the number of years is small, as 20 or under, the rate may be found thus:

The present worth of an annuity may also be conceived to be nearly equal to the area of a geometrical figure, having two of its sides parallel, and two of its angles right; its greatest breadth $= \frac{a}{R}$, its least $= \frac{a}{R^t}$, and its height $= t$; hence $p = \frac{a t}{R^t + 1}$, which gives $\frac{t+1}{2} \sqrt{\frac{at}{p}} = R = 1+r$, nearly.

Ex. 5. Suppose $a = \text{L. } 20$, $p = \text{L. } 220$, and $t = 21$ years. Sought, the rate of interest allowed.

$$\begin{aligned} \text{L. } at, &= 2.6232493 \\ + \text{co. L. } p. &= 7.6575773 \end{aligned}$$

$\frac{t+1}{2} = 11$) 0.2808266 ($.0255297 = 1.0605 = R = 1+r = 6$ per cent. If greater accuracy be desired, let $\frac{6}{t-1} = b$, then $b - \sqrt{bb - 2rb} = r$ corrected. These two cases may also be resolved by the Rule of double position.

Ex. 6. Suppose $a = \text{L. } 20$, $p = \text{L. } 100$, and $t = 7$ years; sought, R .

1st position, $R = 1.09 \dots$ result 100.659... error $+.659$
 2d position, $R = 1.0925 \dots$ result 99.82... error $-.180$

Per question. result 100. diff. of result $= .839$

Hence, $.839 : .0025 :: .180 : .000536 =$ difference between the 2d position and R ; but seeing, in this question, the greater the interest is, *ceteris paribus*, the less will be the result, this difference must be deducted from 1.0925, making $R = 1.091964$; which gives $p = L. 99.9989$.

49. When an annuity is paid every half year, quarterly, or monthly, &c. its present value may be found by the equation $p = a \times 1 - \frac{1}{r} R^t$.

Ex. What is the present worth of an annuity of L. 5 paid every quarter, to continue $20\frac{3}{4}$ years, reckoning interest at $4\frac{1}{2}$ per cent.? Here $a = 5$, $R^t = \overline{1.045}^{20.75}$ and $r = .011065$; hence $5 \times \frac{.598822}{.011065} = L. 270.59$, the present worth.

By comparing this with Ex. 1st, 47. it appears, that an annuity paid every half year or quarterly is more valuable than when paid yearly. See Chap. VIII. Prob. XX.

In like manner may a , t , and R , be found by the other equations of 47th, when the payments are made every half year, quarterly, or monthly.

3d. Of the present Worth of an Annuity in reversion.

50. When an annuity is to commence after a certain number of years, (n) and then to continue for a certain time, (t) its present value is found
by

by subtracting the present worth for n years, from that for $t+n$ years: thus, $a \times \frac{1 - \frac{1}{R^{t+n}}}{r} - a \times \frac{1 - \frac{1}{R^n}}{r} = p$. Or, divide the present value of the annuity at its commencement by R^n , the quotient gives p : Thus, $a \times \frac{1 - \frac{1}{R^t}}{r R^n} = a \times \frac{R^t - 1}{r R^{t+n}} = p$. Hence the following

TABLE.

Given.		
$a, n, t, R,$	$p = a \times \frac{1 - \frac{1}{R^t}}{r R^n} = a \times \frac{1 - \frac{1}{R^t}}{R^n + 1 - R^n}$	Equation 1.
$p, n, t, R,$	$a = r R^n p \times \frac{1}{1 - \frac{1}{R^t}}$	2.
$p, n, t, a,$	$\frac{R^n + 1 - R^n}{1 - \frac{1}{R^t}} = \frac{a}{p}$	3.
$p, R, n, a,$	$t = \frac{L \cdot a - L \cdot a - r R^n p}{L R}$	4.
$p, R, t, a,$	$n = \frac{L \cdot a - \frac{a}{R^t} - L r p}{L, R}$	5.

Ex. 1. There is an yearly annuity of L. 30, which, commencing after the expiration of a lease of 10, is to continue 19 years; it is required to find the present value thereof, reckoning interest at 5 per cent. Here $a = 30$, $n = 10$, $t = 19$, and $R = 1.05$: Hence $p = 30 \times \frac{.604266}{.0814447} = \text{L. } 222.58$, the present value.

Ex. 2. An annuity, which, commencing 12 years hence, is to continue 20 years, being exposed

posed to sale, sold for L. 250. Reckoning interest at 4 per cent. fought the annuity.

Here $p = 250$, $n = 12$, $t = 20$, $R = 1.04$, fought a

$$L. rR^np = 1.2044002$$

$$\text{Equ. 2. } L. \frac{1}{1 - \frac{1}{R^t}} = .2647102$$

$$1.4691104 = L. 29.4517, \text{ the ann.}$$

Ex. 3. When 12 years of a lease of 21 years were expired, a renewal for the same term was granted for L. 500; 8 years are now expired, and for what sum must a corresponding renewal be made, reckoning 5 per cent. compound interest? Here,

1st, Given $p = 500$, $n = 9$, $t = 12$, $R = 1.05$, fought a

$$\text{Equ. 2. } p \times \frac{rR^t + n}{R^t - 1} = 500 \times \frac{.139298}{.79585} = L. 87.5, \text{ the ann.}$$

2. Given $a = 87.5$, $n = 13$, $t = 8$, fought p .

$$\text{Equ. 1. } a \times \frac{R^t - 1}{rR^t + n} = 87.5 \times \frac{.47746}{.139298} = L. 299.9, \text{ the fine.}$$

When a , p , n , and t are given, R may be found by approximation: Thus, let $T = 2n + t + 1$, then $\frac{T}{2} \sqrt{\frac{at}{p}} = R = 1 + r$ nearly. If greater exactness be desired, let $\frac{6T+6}{n} = b$, then $b - \sqrt{bb - 2rb}$ is exceeding near the value of r .

Ex. 4th. An annuity of L. 40 per annum, being in possession for the term of 21 years, for L. 80 paid down can be prolonged for 10 years more, or to 31 years; what is the rate of interest required? Here $a = 40$, $p = 80$, $n = 21$, and $t = 10$, fought R .

L. 4

$$\begin{aligned}
 L. at &= 2.6020600 & T &= 42 + 11 = 53 \\
 ex. L. p &= 8.0969100 \\
 \frac{T}{2} &= 26.5 \quad 0.6989700 \quad (0.0263762 = L. 1 + r = 1.062616. \\
 b &= \frac{6T+6}{tt} = 3.24 & 2r &= .125232. \\
 b - 2r &\times b = 3.114768 \times 3.24 = 10.091848 = bb - 2rb. \\
 b &= 3.24 \\
 \sqrt{bb - 2rb} &= 3.176767 \\
 .063233 &= r, \text{ the rate sought.}
 \end{aligned}$$

Ex. 5. Given $a = L. 87.5$, $p = L. 500$, $n = 9$ years, $R = 1.05$, sought t , the time of the annuity's continuance.

$$\text{Equ. 4. } \frac{.2542796}{.0211893} = 12 \text{ years.}$$

Ex. 6. Given $a = L. 20$, $p = L. 145.836$, $t = 12$ years, $R = 1.05$, sought n , the number of years prior to the commencement of the annuity.

$$\text{Equ. 5. } \frac{.0847572}{.0211893} = 4 \text{ years.}$$

of

Of the Renewing of Leases for a certain Number of Years,

A TABLE, shewing the Fine for renewing of any Number of Years elapsed in a Lease of 21 Years.

Years elapsed.	L. 11 : 11 : 81 per cent.	5 per cent.	6 per cent.	8 per cent.	10 per cent.
1	0.1000	0.3590	0.2942	0.1987	0.1352
2	0.2118	0.7359	0.6060	0.4122	0.2848
3	0.3364	1.1317	0.9365	0.6489	0.4473
4	0.4754	1.5471	1.2869	0.8942	0.6272
5	0.6306	1.9834	1.6583	1.1544	0.8250
6	0.8035	2.4415	2.0419	1.4563	1.0426
7	1.0000	2.9226	2.4691	1.7726	1.2820
8	1.2121	3.4276	2.9114	2.1120	1.5454
9	1.4526	3.9576	3.3803	2.4807	1.8350
10	1.7210	4.5148	3.8772	2.8779	2.1437
11	2.0205	5.0995	4.4040	3.3067	2.5042
12	2.3446	5.7134	4.9624	3.7699	2.8897
13	2.7275	6.3580	5.5543	4.2702	3.3038
14	3.1435	7.0349	6.1817	4.8104	3.7803
15	3.6077	7.7456	6.7468	5.3939	4.2935
16	4.1258	8.4917	7.5417	6.0241	4.8579
17	4.7038	9.2752	8.2990	6.7047	5.4789
18	5.3488	10.0980	9.0911	7.4397	6.1619
19	6.0686	10.9618	9.9207	8.2336	6.9132
20	6.8716	11.8678	10.8107	9.0809	7.7396
21	7.7682	12.8212	11.7641	10.0168	8.6487

{ Total value.

In the construction of this table, let n be the number of years in *esse*, or yet to run by the lease, t the time elapsed since its commencement; then from the present value of L. 1 annuity for

L $n+t$

$n+t$ years, at a given rate, subtract that of an annuity for n years; the remainder $= N$, the tabular number expressing the fine for renewing the lease; the yearly rent being L. 1.

Ex. There is a lease of church-lands, for the space of 21 years, at L. 70 per annum, whereof 14 years are expired: It is required to know what fine ought to be paid, to have the same renewed, reckoning interest at L. 11 : 11 : $8\frac{1}{4}$ per cent.

Let a express the rent, and N the tabular number for 14 years; then

$$aN = 70 \times 3.1435 = \text{L. } 220.045, \text{ the fine required.}$$

When the length of the lease is different from 21, suppose 31 years, whereof t years are elapsed: In this case, from the present value of L. 1 annuity for $n+t$ ($= 31$) years, at a given rate, Tab. V. subtract that of an annuity for n years yet to run; the remainder will shew the fine for renewing the lease, supposing the rent to be L. 1.

When the fine for the whole lease is a given sum, for instance L. 500; see Ex. 3d; find a , the annuity which that sum will purchase, at a given rate, during the currency of the lease; then a multiplied by N , corresponding to the years elapsed, will produce the fine.

4th, Of the present Worth of the Reversion of an Annuity.

51. The reversion of an annuity being the same with a reversion in fee-simple, which is to commence after a certain time, t , $=$ the duration of the annuity, its present value may be found by subtracting

tracting the value of the annuity from that of the perpetuity, the remainder gives the reversion: Or rather thus, Divide the present value of the perpetuity, viz. $\frac{a}{r}$ (56.) by R^t , the quotient gives the reversion; thus $\frac{a}{rR^t} = p$, the reversion.

Ex. 1st, Sought the reversion of an yearly annuity of L. 20, which is to continue 12 years, reckoning interest at 5 per cent.

$$\frac{a}{rR^t} = \frac{20}{.0897925} = \text{L. } 222.735, \text{ the reversion.}$$

Ex. 2^d, Sought, the reversion of an yearly annuity of L. 40, which is to commence 21 years hence, and to continue 10 years, reckoning interest at 6 per cent. Here $a=40$, $n+t=31$, and $R=1.06$.

$$\frac{a}{rR^{n+t}} = \frac{40}{.365286} = \text{L. } 109.5, \text{ the reversion.}$$

See reversions more fully treated of in 57.

52. The several particulars respecting annuities at compound interest may more readily be found by the help of the three following tables, wherein are calculated the amount and present value of L. 1 annuity for years, at several rates of compound interest.

TABLE IV. Shewing the Amount of L. 1 Annuity for Years.

Yrs.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1	1.	1.	1.	1.	1.	1.
2	2.03	2.035	2.04	2.045	2.05	2.06
3	3.0909	3.10623	3.1216	3.13703	3.1525	3.1836
4	4.18363	4.21494	4.24646	4.27819	4.31013	4.37460
5	5.30913	5.36246	5.41632	5.47071	5.52563	5.63709
6	6.46841	6.55015	6.63297	6.71689	6.80191	6.97532
7	7.66246	7.77941	7.89829	8.01915	8.14201	8.39384
8	8.89233	9.05168	9.21423	9.38001	9.54911	9.89747
9	10.15911	10.36849	10.58279	10.80211	11.02656	11.49131
10	11.46388	11.73139	12.00611	12.28821	12.57789	13.18079
11	12.80779	13.14199	13.48635	13.84118	14.20678	14.97164
12	14.19203	14.60196	15.02580	15.46403	15.91712	16.86994
13	15.61779	16.11303	16.62684	17.15991	17.71298	18.88214
14	17.08632	17.67698	18.29191	18.93211	19.59863	21.01506
15	18.59891	19.29568	20.02358	20.78405	21.57856	23.27597
16	20.15688	20.97103	21.82453	22.71933	23.65749	25.67253
17	21.76158	22.70501	23.69751	24.74171	25.84036	28.21288
18	23.41443	24.49969	25.64541	26.85508	28.13238	30.90565
19	25.11687	26.35718	27.67123	29.06356	30.53900	33.75999
20	26.87037	28.27968	29.77810	31.37142	33.06595	36.78559
21	28.67648	30.26947	31.96920	33.78313	35.71925	39.99273
22	30.53678	32.32890	34.24797	36.30338	38.50521	43.39229
23	32.45288	34.46041	36.61789	38.93703	41.43047	46.99583
24	34.42647	36.66653	39.08260	41.68919	44.50199	50.85557
25	36.45926	38.94985	41.64591	44.56521	47.72709	54.86451
26	38.55304	41.31310	44.31174	47.57064	51.11345	59.15638
27	40.70963	43.75906	47.08421	50.71132	54.66913	63.70576
28	42.93092	46.29063	49.96758	53.99333	58.40258	68.52811
29	45.21885	48.91080	52.96628	57.42303	62.32271	73.63979
30	47.57541	50.62267	56.08494	61.00707	66.43884	79.05818
31	50.00268	54.42947	59.32833	64.75238	70.76079	84.80167
32	52.50276	57.33450	62.70147	68.66624	75.29883	90.88977
33	55.07784	60.34121	66.20953	72.75623	80.06377	97.34316
34	57.73017	63.45315	69.85791	77.03025	85.06696	104.18375
35	60.46208	66.67401	73.65222	81.49662	90.32031	111.43478
36	63.27594	70.00760	77.59831	86.16396	95.83632	119.12086
37	66.17422	73.45787	81.70224	91.04134	101.62814	127.26812
38	69.15945	77.02889	85.97033	96.13820	107.70954	135.90420
39	72.23423	80.72490	90.40915	101.46442	114.09502	145.05845
40	75.40126	84.55028	95.02551	107.03032	120.79977	154.76196
41	78.66330	88.50954	99.82654	112.84668	127.83976	165.04767
42	82.02320	92.60737	104.81959	118.92478	135.23175	175.95053
43	85.48390	96.84863	110.01238	125.27640	142.99333	187.50756
44	89.04841	101.23816	115.41287	131.91384	151.14300	199.75801
45	92.71986	105.78167	121.02939	138.84996	159.70015	212.74349
46	96.50146	110.48403	126.87056	146.09821	168.68516	226.50810
47	100.39650	115.35097	132.94539	153.67263	178.11942	241.09858
48	104.40839	120.38825	139.26320	161.58790	188.02539	256.56449
49	108.54065	125.60184	145.83373	169.85935	198.42666	272.95836
50	112.79686	130.99791	152.66708	178.50303	209.34799	290.33586

Table IV. Shews the amount of L. 1 annuity; and is constructed by Equation 1st, 44; whereby

$$N = 1 \times \frac{R^t - 1}{r}. \text{ Or thus, seeing}$$

$$\begin{aligned} 1 &= a = \text{amount for 1 year.} \\ a + R &= b = \text{for 2 years.} \\ b + R^2 &= c = \text{for 3 years.} \\ c + R^3 &= d = \text{for 4 years, \&c.} \end{aligned}$$

Hence, to L. 1, the first year of this table, add the first year of Tab. II. their sum is the second year of this table, to which add the second year of Tab. II. their sum will be the third year of this table, &c. From the nature of the table, $1 : N :: a : Na = s$, the amount of any annuity a , for the time and at the rate corresponding to N . Hence the following

T A B L E.

Given.		
$a, t, r,$	$s = Na.$	r being given, t is found opposite to N , and t being given, r is found above N in the Table.
$s, t, r,$	$a = \frac{s}{N}.$	
$s, t, a,$	$N = \frac{s}{a}.$	
$s, r, a,$		

Ex. 1st. Given, $a = \text{L. } 80$, $t = 20$ years, and the rate 4 per cent.; sought the amount. Here,

$$Na = 29.7781 \times 80 = \text{L. } 2382.248, \text{ the amount.}$$

Ex. 2d. Given, $s = \text{L. } 1000$, $t = 15$ years, and the rate of interest 5 per cent. ; fought, the annuity.

$$\frac{s}{N} = \frac{1000}{21.57856} \doteq \text{L. } 46.3423, \text{ the annuity.}$$

Ex. 3d. Given, $a = \text{L. } 48.114$, $s = \text{L. } 1000$, and $t = 15$ years; fought, the rate of interest.

$$\frac{s}{a} = \frac{1000}{48.114} = 20.7840 = N \text{ for } 15 \text{ years, corresponding to } 4\frac{1}{2} \text{ per cent.}$$

Ex. 4th. Given $a = \text{L. } 5.25$, $s = \text{L. } 419.297$, and the rate $4\frac{1}{2}$ per cent. ; fought, the time of continuance.

$$\frac{s}{a} = \frac{419.297}{5.25} = 79.8661 = N \text{ for } 4\frac{1}{2} \text{ per cent. corresponding to } 34 \text{ years; and for the odd days say,}$$

$$\begin{array}{r} 81.49662 \\ 77.03025 \\ \hline \end{array} \quad \begin{array}{r} 79.86610 \\ 77.03025 \\ \hline \end{array}$$

$$\text{as } 4.46637 : 1 :: 2.83585 : .635 = 232 \text{ days.}$$

TABLE

TABLE V. Shewing the present Worth of
L. 1 Annuity for Years.

Yrs.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1	0.97087	0.96618	0.96154	0.95694	0.95238	0.94339
2	1.91347	1.89969	1.88609	1.87267	1.85941	1.83339
3	2.82861	2.80164	2.77509	2.74896	2.72325	2.67301
4	3.71710	3.67308	3.62989	3.58752	3.54595	3.46510
5	4.57971	4.51505	4.45182	4.38997	4.32947	4.21236
6	5.41719	5.32855	5.24214	5.15787	5.07569	4.91732
7	6.23028	6.11454	6.00205	5.89270	5.78637	5.58238
8	7.01969	6.87395	6.73274	6.59588	6.46321	6.20979
9	7.78611	7.60768	7.43533	7.26879	7.10782	6.80169
10	8.53020	8.31660	8.11089	7.91272	7.72173	7.36008
11	9.25262	9.00155	8.76047	8.52892	8.30641	7.88687
12	9.95400	9.66333	9.38507	9.11858	8.86325	8.38384
13	10.63495	10.30274	9.98565	9.68285	9.39357	8.85268
14	11.29607	10.92052	10.56312	10.22282	9.89264	9.29498
15	11.93793	11.51741	11.11838	10.73954	10.37966	9.71225
16	12.56110	12.09412	11.65229	11.23401	10.83776	10.10589
17	13.16612	12.65132	12.16567	11.70719	11.27406	10.47726
18	13.75351	13.18968	12.65929	12.15999	11.68958	10.82760
19	14.32380	13.70983	13.13394	12.59329	12.08532	11.15812
20	14.87747	14.21140	13.59032	13.00793	12.46221	11.46992
21	15.41502	14.69797	14.02916	13.40472	12.82115	11.76407
22	15.93691	15.16712	14.45111	13.78442	13.16300	12.04158
23	16.44360	15.62041	14.85684	14.14777	13.48857	12.30338
24	16.93554	16.05836	15.24696	14.49548	13.79864	12.55036
25	17.41315	16.48151	15.62208	14.82821	14.09394	12.78335
26	17.87684	16.89035	15.98276	15.14661	14.37518	13.00316
27	18.32703	17.28536	16.32958	15.45130	14.64303	13.21053
28	18.76411	17.66702	16.66306	15.74287	14.89813	13.40616
29	19.18845	18.03576	16.98371	16.02189	15.14107	13.59072
30	19.60044	18.39204	17.29203	16.28888	15.37245	13.76483
31	20.00043	18.73627	17.58849	16.54439	15.59281	13.92910
32	20.38876	19.06886	17.87355	16.78889	15.80267	14.08404
33	20.76579	19.39020	18.14764	17.02286	16.00255	14.23023
34	21.13183	19.70068	18.41119	17.24676	16.19290	14.36814
35	21.48722	20.00066	18.66461	17.46102	16.37419	14.49824
36	21.83225	20.29049	18.90828	17.66604	16.54685	14.62098
37	22.16723	20.57052	19.14258	17.86224	16.71128	14.73678
38	22.49246	20.84108	19.36786	18.04999	16.86789	14.84602
39	22.80821	21.10250	19.58448	18.22965	17.01704	14.94907
40	23.11477	21.35507	19.79277	18.40158	17.15908	15.04629
41	23.41240	21.59910	19.99305	18.56611	17.29436	15.13802
42	23.70136	21.83488	20.18562	18.72355	17.42320	15.22454
43	23.98190	22.06269	20.37079	18.87421	17.54591	15.30617
44	24.25427	22.28279	20.54884	19.01838	17.66277	15.38318
45	24.51871	22.49545	20.72004	19.15634	17.77406	15.45583
46	24.77545	22.70092	20.88465	19.28837	17.88006	15.52437
47	25.02471	22.89943	21.04293	19.41471	17.98101	15.58902
48	25.26671	23.09124	21.19513	19.53560	18.07715	15.65002
49	25.50166	23.27656	21.34147	19.65130	18.16872	15.70757
50	25.72976	23.45562	21.48218	19.76201	18.25592	15.76186

Table V. Shews the present value of L. 1 annuity, and is constructed by Equation 1st, 47.

whereby $N = 1 \times \frac{1 - \frac{1}{R^t}}{r}$ = the number of years purchase an annuity is worth: Or thus, Seeing

$$\frac{1}{R} = a = \text{Present worth for 1 year.}$$

$$a + \frac{1}{R^2} = b = \text{for 2 years.}$$

$$b + \frac{1}{R^3} = c = \text{for 3 years.}$$

$$c + \frac{1}{R^4} = d = \text{for 4 years, \&c.}$$

Hence the first year in this table and in table III. are the same, the sum of the first year in this and of the second in Table III. makes the second in this, and the sum of the second in this, and of the third in Table III. makes the third year in this table, &c. From the nature of the table, $1 : N :: a : Na = p$, the present worth of any annuity a , for the time, and at the rate corresponding to N . Hence the following

T A B L E.

Given.	
$a, t, r,$	$p = Na$
$p, t, r,$	$a = \frac{p}{N}$
$p, t, a, \}$	$N = \frac{p}{a}$
$p, r, a, \}$	

Ex. 1st, Given $a = \text{L. } 50$, $t = 15$ years, and the rate 5 per cent.; sought the present worth. Here,

$$Na = 10.37966 \times 50 = \text{L. } 518.983 = p.$$

Ex.

Ex. 2d, Given, $p=270$, $t=50$, and the rate 3 per cent. ; fought, the annuity. Here,

$$\frac{p}{N} = \frac{270}{25.729764} = L. 10.493685, \text{ the annuity.}$$

Ex. 3d, Given, $a=L. 100$, $p=L. 1037.966$, and $t=15$ years ; fought, the rate of interest.

$$\frac{p}{a} = \frac{1037.966}{100} = 10.37966 = N \text{ for } 15 \text{ years, corresponding to } 5 \text{ per cent.}$$

Thus, as far as the tables extend, may these difficult equations be resolved, where R is fought in Equation 3d, 44. and 47. and where it is involved to a high power.

M

TABLE

TABLE VI. Shewing the Annuity which L. 1 will purchase.

Yrs.	3 per cent	per cent	4 per cent.	per cent.	5 per cent	6 per cent.
1	1.03	1.035	1.04	1.045	1.05	1.06
2	.522610	.526400	.530196	.533997	.537805	.545437
3	.353530	.356934	.360348	.363773	.367208	.374109
4	.269027	.272251	.275490	.278743	.282011	.288591
5	.218354	.221481	.224627	.227791	.230974	.237396
6	.184597	.187668	.190762	.193878	.197016	.203362
7	.160506	.163544	.166609	.169701	.172820	.179135
8	.142456	.145476	.148527	.151609	.154722	.161036
9	.128434	.131446	.134493	.137574	.140690	.147022
10	.117230	.120241	.123291	.126378	.129504	.135868
11	.108077	.111092	.114149	.117248	.120389	.126793
12	.100462	.103484	.106552	.109666	.112825	.119277
13	.094029	.097061	.100143	.103275	.106455	.112960
14	.088526	.091570	.094669	.097820	.101024	.107584
15	.083766	.086825	.089941	.093114	.096342	.102962
16	.079611	.082684	.085820	.089015	.092269	.098952
17	.075952	.079043	.082198	.085417	.088699	.095444
18	.072708	.075816	.078993	.082237	.085546	.092356
19	.069813	.072940	.076138	.079407	.082745	.089620
20	.067215	.070361	.073582	.076876	.080242	.087184
21	.064871	.068036	.071280	.074600	.077996	.085004
22	.062747	.065932	.069198	.072545	.075970	.083045
23	.060813	.064018	.067309	.070682	.074136	.081278
24	.059047	.062273	.065586	.068987	.072471	.079679
25	.057427	.060674	.064012	.067439	.070952	.078226
26	.055938	.059205	.062567	.066021	.069564	.076904
27	.054564	.057852	.061238	.064719	.068292	.075697
28	.053329	.056602	.060013	.063521	.067122	.074592
29	.052114	.055445	.058879	.062414	.066045	.073579
30	.051019	.054371	.057830	.061391	.065051	.072649
31	.049999	.053372	.056855	.060443	.064132	.071752
32	.049046	.052441	.055948	.059563	.063280	.071002
33	.048156	.051572	.055104	.058744	.062490	.070273
34	.047322	.050759	.054315	.057982	.061755	.069598
35	.046539	.049998	.053577	.057270	.061072	.068974
36	.045804	.049284	.052887	.056606	.060434	.068395
37	.045111	.048613	.052239	.055984	.059840	.067857
38	.044459	.047982	.051632	.055402	.059284	.067358
39	.043844	.047387	.051061	.054857	.058764	.066894
40	.043262	.046827	.050523	.054343	.058278	.066462
41	.042712	.046298	.050017	.053861	.057822	.066059
42	.042191	.045798	.049540	.053408	.057395	.065683
43	.041698	.045325	.049090	.052982	.056993	.065333
44	.041230	.044877	.048664	.052581	.056616	.065006
45	.040785	.044453	.048262	.052202	.056262	.064701
46	.040362	.044051	.047882	.051845	.055928	.064415
47	.039960	.043669	.047522	.051507	.055614	.064148
48	.039577	.043306	.047181	.051188	.055318	.063927
49	.039213	.042962	.046857	.050887	.055039	.063664
50	.038865	.042634	.046550	.050602	.054777	.063444

Table VI. shews the annuity which L. 1 will purchase, and is constructed by Equation 2d, 47. whereby $N = \frac{r}{1 - \frac{1}{R^t}}$. Or thus, it being the reci-

procal of Table V. N of Table VI. $= \frac{1}{N}$ of Table V. From the nature of the table, $1 : N :: p : Np = a$, the annuity which p will purchase, for the time and at the rate corresponding to N . Hence the following

T A B L E.

$p, t, R,$	$a = Np.$
$a, t, R,$	$p = \frac{a}{N}.$
$p, t, a, \}$	$N = \frac{a}{p}.$
$p, R, a, \}$	

Ex. 1st, Given, $p = \text{L. } 1000$, $t = 20$ years, and the rate 5 per cent. ; sought, the annuity. Here,

$$Np = .080242 \times 1000 = \text{L. } 80.242, \text{ annuity.}$$

Ex. 2d, Given, $a = \text{L. } 30$, $t = 19$ years, and the rate 4 per cent. ; sought, the present worth. Here,

$$\frac{a}{N} = \frac{30}{.076138} = \text{L. } 394.021, \text{ the present worth.}$$

Ex. 3d, Given, $a = \text{L. } 80.242$, $p = \text{L. } 1000$, and $t = 20$ years ; sought, the rate of interest. Here,

$$\frac{a}{p} = \frac{80.242}{1000} = .080242 = N \text{ for 20 years, corresponding to 5 per cent. the rate sought.}$$

More Examples.

1st, A gentleman purchases an estate for L. 1000, and is allowed to retain the purchase-money in his own hands, upon condition of paying to the former proprietor 4 per cent. per annum: He sets the estate at such a yearly rent as to have 6 per cent for his money; sought, the time in which the estate will clear itself.

Here the purchaser's yearly profit-rent is 2 per cent. upon the purchase-money, and the time is sought in which this interest will equal the principal.—By Remark 3d, 43,

$$s = 2p \text{ when } t = \frac{L. 2}{L. 1.02} = \frac{.30103}{.0086} = 34.886 \text{ years.}$$

2d, A puts L. 1500 out at interest, B hath an annuity of L. 75 to continue 50 years; which of them will amount to the greatest sum at the end of the 50 years; and what is the present worth of the difference, at the rate of $4\frac{1}{2}$ per cent. per annum? Here, by Table II. and IV.

The amount of L. 1500 for 50 years at $4\frac{1}{2}$ per cent. = L. 13548.945
 of L. 75 annuity for 50 years = 13387.727

A's amount exceeds that of B by 161.218

The present worth of which by Table III. = L. 17.848.

3d, A hath a term of 7 years in an estate of L. 50 per annum; B hath a reversion of the same estate of 14 years after A; and C hath a further reversion of 21 years after A and B; sought, the present worths of the several terms, reckoning

interest at $4\frac{1}{2}$ per cent. Here $a \times \frac{1 - \frac{1}{R^t}}{rR^u}$ = their present worths respectively, Or thus,

By

By Table V. 42 years = $18.72355 \times 50 = 936.1775$

21 years = $13.40472 \times 50 = 670.236$

7 years = $5.89270 \times 50 = 294.635$

subtracting these from each preceding term, we have

A's term = L. 294.635, B's = L. 375.601, C's = L. 265.9415.

4th, A person having 7 years to run in a lease of an estate of L. 80 per annum, it is required what sum he ought to pay at present to have the lease renewed by having 20 years added thereto, reckoning interest at 6 per cent. ? Here,

$$a \times \frac{1 - \frac{1}{R^{20}}}{rR^7} = 80 \times 7.62815 = \text{L. } 610.25 = p. \text{ Or thus,}$$

p , of L. 1 annuity for 27 years = 13.21053

for 7 years = 5.58238

$$\underline{7.62815 \times 80 =}$$

L. 610.25, = the sum he ought to pay.

5th, A lease of an estate, to continue 14 years, is offered for L. 250 fine, and L. 44 yearly rent; but the tenant wants to reduce the rent to L. 20 per annum; what ought the fine to be at 6 per cent? Here,

$$44 - 20 = 24, \text{ and Tab. V. } 9.29498 \times 24 = 223.0795$$

$$+ 250.$$

The whole fine = L. 473.0795

6th, The proprietor of an estate, having, 20 years ago, granted the lease of a farm for 38 years, is desirous of recovering the same: It is required to find what he ought to pay for the recovery of said

faid lease, upon the supposition that the farm is worth L. 40 of yearly additional rent, and reckoning interest at 5 per cent. Here are given, $a = \text{L. } 40$, $t = 18$, and the rate 5 per cent. sought p .

By Tab. V. $11.6896 \times 40 = \text{L. } 467.584 = p$.

53. *Of Annuities computed at Compound Interest, when the first Payment is either due, or paid per Advance.*

1st, To find the amount of an annuity in arrears, the first payment being due per advance, at a given rate of compound interest. Here, as the first year's annuity (a) bears interest, the last year's annuity will be represented by aR^t ; and deriving a series from the quantity aR , in the ratio of $1 : R$, we shall have $aR + aR^2 + aR^3 \dots + aR^t = s$; hence $s = a \times \frac{R^{t+1} - R}{r} = a \times \frac{R^t - 1}{r} \times R = aN \times R$ of Table IV.

Ex. It is required, to find the amount of an annuity of L. 5, at the end of 30 years, the first payment being due per advance, and reckoning interest at 4 per cent. Here,

$$a \times \frac{R^{t+1} - R}{r} = 5 \times \frac{2.33373}{.04} = \text{L. } 291.64, \text{ the amount.}$$

If the series expressing the amount of an annuity, when the first payment is due at the end of the year, (44.) be subtracted from the above series, their difference will be $aR^t - a$, in favour of the latter, which, of consequence, will exceed the former, by the accumulated interest of one year's annuity for the time of its continuance. Or thus: The amount of an annuity for t years, when the first payment is due from the beginning of the year,

year, will be equal to its amount for $t+1$ years— a , when the first payment is due at the end of the year.

2d, To find the present value of an annuity, the first payment being made *per* advance, at a given rate of compound interest. Here we shall have,

$a + \frac{a}{R} + \frac{a}{R^2} \dots + \frac{a}{R^{t-1}} = p$, the present value of the annuity, which gives $p = a \times \frac{R - \frac{1}{R^{t-1}}}{R - 1} = a \times \frac{1 - \frac{1}{R^t}}{R - 1} \times R = aN \times R$ of Table V.

Ex. It is required, to find the present value of an annuity of L. 20, to continue 19 years, the first payment being made *per* advance, and reckoning interest at 4 per cent. Here,

$$a \times \frac{R - \frac{1}{R^{t-1}}}{R - 1} = 20 \times \frac{.5464}{.04} = \text{L. } 273.4, \text{ the value.}$$

If the series expressing the present value of an annuity, when the first payment is made at the end of the year, (47.) be subtracted from the above series, their difference will be $a - \frac{a}{R^t}$, in favour of the latter, which, of consequence, will exceed the former by the difference betwixt the annuity and the present value of one payment thereof due at the end of t years. Or thus: The present value of an annuity for t years, when the first payment is made *per* advance, will be equal to its present value for $t-1$ years $+a$, when the first payment is due at the end of the year.

Remarks

Remarks upon Annuities at Compound Interest.

54. 1st, The expression $\frac{1 - \frac{1}{R^t}}{r}$, in Equation 1st, 47. is always less than $\frac{1}{r}$, except when t , the time, is indefinite; of consequence, the years purchase of an annuity for a certain number of years, at compound interest, are always less than the years purchase of a perpetuity: Hence compound interest, by which the value of a perpetuity is determined, is the proper standard for ascertaining the value of an annuity.

2d, The time in which the present value of an annuity is equal to its reversion may be found by the equation $\frac{a}{r} - \frac{a}{rR^t} = \frac{a}{rR^t}$, which gives $\frac{a}{r} = \frac{2a}{rR^t}$.

$$1 = \frac{2}{R^t}.$$

$$R^t = 2, \text{ and}$$

$t = \frac{L. 2}{L. R}$ = the time in which a sum of money doubles itself at a given rate of compound interest.

3d, Besides the two methods mentioned in the construction of Tables IV. and V. the amount and present value of an annuity may also be calculated thus:

1st, To find the amount of L. 1 annuity for years.

Let $1 = a$ = the amount for 1 year.

$$\text{Then } aR + 1 = b =$$

2.

$$bR + 1 = c =$$

3.

$$cR + 1 = d =$$

4 years, &c.

2d,

2d, To find the present value of L. 1 annuity for years.

Let $\frac{1}{R} = a$ = the present value for 1 year.

Then $\frac{a+1}{R} = b =$ 2.

$\frac{b+1}{R} = c =$ 3.

$\frac{c+1}{R} = d =$ 4 years, &c.

SUPPLEMENT to CHAPTER VI.

Of a Sinking Fund to extinguish the National Debt.

55. **W**HEN a debtor's funds are insufficient to pay the interest of his debt, this must increase yearly by the accumulation of the deficient interest; but if he can spare a little, after paying this interest, that little may, with proper œconomy, become a sinking fund to extinguish his debt in time.

Any part of the national debt is said to be bought at par, when the legal interest of the purchase money, at 5 per cent. is equal to the interest of the stock purchased, or when the purchaser hath 5 per cent for his money.

Thus, pr cents. L. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6,
Pars, L. 60, 70, 80, 90, 100, 110, 120.

As a debtor hath it always in his power to discharge his debt, by paying the sum borrowed, when he is able, and is so disposed, Government, it is presumed, is not obliged to pay above the par of 5 per cent for each L. 100 of national debt.—

In the following particulars, units are assumed in place of millions, when the national debt is concerned; which debt may consist either of money borrowed at 3, 4, or 5 per cent. or in the value of annuities paid by Government, for a certain number of years, at 5 or 6 per cent.; and may be in all about L. 262, at $3\frac{1}{2}$ per cent. *anno* 1792; at which period there was, in the space of six years, the sum of L. 8 recovered, and the national sinking fund was raised to L. 1.4, which is supposed to accumulate till it rises to L. 4.

1st, It is required, to find the time in which a sinking fund of L. 1.4 will extinguish L. 25—8 = L. 17 of 3 per cents, the purchases being made at the market-price of 90.

Here, $100 : 90 :: 17 : 15.3$; and, by Table IV.

$$\frac{s}{a} = \frac{15.3}{1.4} = 10.928 = N, \text{ at 3 per cent. corresponding to } 9.59 \text{ years, the time required; being that period at which Government can, if it shall be thought proper, point all the artillery of the sinking fund against the 4 and 5 per cents, for the reduction of the same.}$$

2d, It is required, to find the time in which the national sinking fund will rise to L. 4; beyond which it is supposed it needs not to accumulate, being then sufficient to answer all the purposes of its destination.

On or before the year 1808, to which period the long annuities of 1778 extend, it is presumed that L. 12 of annuities, at 5 per cent. will cease whose interest = L. .60.

Therefore, to the present sinking fund = 1.40
Add the interest of L. 17 recovered in 9.59 years = 0.51

Their sum = L. 1.91

And

And find the capital of 4 per cents, which a sinking fund of L. 1.91 will recover in $16 - 9.59 = 6.41$ years. $Na = 7.152 \times 1.91 = \text{L. } 13.66$, whose interest, at 4 per cent.

$=$	.546
to which add $1.91 + .60$	$= 2.510$

Their sum is the sinking fund, an. 1808 $= 3.056$

Again, $4 - 3.056 = .944$; and $\frac{.244}{.04} = \text{L. } 23.6$, a capital of 4 per cents, which L. 3.056 will recover in the space of 7 years. Hence the national sinking fund will rise to L. 4 anno 1815.

3d, It is required, to find the amount of the national debt discharged when the sinking fund rises to L. 4.

Prior to an. 1792, there was redeemed, in 6 yrs, a sum	$= \text{L. } 8.00$
From hence, in the space of 9.59 years, a sum	$= 17.00$
During 6 41 years, prior to anno 1808,	$= 13.66$
In anno 1808, of annuities,	$= 12.00$
During seven years, prior to anno 1815,	$= 23.60$

In all, L. 74.26

That a sinking fund may have its proper effect, it ought to accumulate yearly, with the interest of the capital which it hath recovered. But when it is limited, for instance, to L. 4, the nation may be relieved from taxes to the amount of said interest.

4th, It is required, to find the time in which a sinking fund of L. 4, without accumulation, will reduce the national debt to the same state in which it was anno 1755, the medium rate of interest being $3\frac{1}{2}$ per cent. and the purchases made at 100.

Let L. 9.45 be the interest of the national debt, anno 1786, corresponding to a capital of L. 270,

N 2

at

at $3\frac{1}{2}$ per cent. Upon this supposition :

To the amount of debt anno 1755, = L. 72.00

Add of capital recovered in 29 years

prior to anno 1815, as above, a sum = 74.26

And of annuities expiring, = 10.00

Their sum = L. 156.26

Hence $270 - 156.26 = L. 113.74$, the debt to be extinguished; sought, the time. Here,

$\frac{113.74}{4} = 28.43$ years, the time required, extending to the year 1844: at which period the national taxes, including one million of a sinking fund, will be $10.45 - 2.52 = 7.93$ millions less, yearly, than they were anno 1786.

5th, It is required, to find the effect which a sinking fund of L. 1.4 will have upon a debt of L. 270, in the space 25 years, reckoning interest at $3\frac{1}{2}$ per cent.; and what will be the amount of said fund at the expiration of that period.

By Table IV. $Na = 38.95 \times 1.4 = L. 54.53$, the debt discharged in 25 years; and $1.4 + 1.9 = L. 3.3$, the sinking fund.

6th, It is required, to find the time in which a sinking fund of L. 1.4 will extinguish a debt of L. 270, reckoning interest at $3\frac{1}{2}$ per cent. Here,

By Equation 4. $44. \frac{L. \frac{n}{a} + 1}{L. R} = \frac{.8893017}{.0149403} = 59.52$ years, the time required,

C H A P. VII.

Of the Value of a Perpetuity, or Freehold Estate.

56. **I**N treating of a perpetuity or freehold-estate; let s represent the value, and a the yearly rent thereof; then the progression indefinitely continued, viz. $\frac{a}{R} + \frac{a}{R^2} + \frac{a}{R^3} + \frac{a}{R^4}$, &c. $= s$. (47.) which progression, having no last term, gives $s = \frac{a}{R-1} = \frac{a}{r}$. (40.) from whence all the varieties in the buying and selling of estates in fee-simple may be resolved, as in the following

T A B L E.

Given.		Sought s .	
$a, R,$	$s = \frac{a}{R-1} = \frac{a}{r}$.	38.	$s = s - \frac{a}{R} \times R.$ $s = Rs - a.$ $\overline{R-1} \times s = a.$ Or thus:
$s, r,$	$a = rs.$		
$s, R,$	$r = \frac{a}{s} \text{ \& } R = 1 + \frac{a}{s}$	40.	$s = \frac{R \times \frac{a}{R}}{R-1} = \frac{a}{R-1}.$

Let

Let N represent the number of years purchase a is worth, then $Na = s$; and for a substituting its value, rs , from the fee-simple equation, we shall have $Nrs = s$; which gives $Nr = 1$, $N = \frac{1}{r}$ & $r = \frac{1}{N}$. Or thus: Seeing $s = Na$, and a , the rent, may be considered as the interest drawn yearly for the purchase-money, hence $s : a = Na : a = N : 1 = 1 : r$, which give $s = Na$, $rs = a$, and $Nr = 1$, from whence the following

EQUATIONS.

$$\begin{aligned} s &= Na = \frac{a}{r}. \\ a &= \frac{s}{N} = rs. \\ N &= \frac{s}{a} = \frac{1}{r}. \\ r &= \frac{a}{s} = \frac{1}{N}. \end{aligned}$$

Ex. 1st, Sought, the value of a freehold estate, whose yearly rent is L. 150, discounting interest at $3\frac{1}{2}$ per cent. Here $\frac{a}{r} = \frac{150}{.035} = \text{L. } 4285.7 = s$, the value. Had the purchase been made at 28.57 years purchase, corresponding to $3\frac{1}{2}$ per cent. then $150 \times 28.57 = \text{L. } 4285.5 = s$, the value.

Ex. 2d, A gentleman purchases an estate for L. 14,000; at what yearly rent must he let it, to have 4 per cent. upon his purchase-money? Here, $rs = .04 \times 14000 = \text{L. } 560 = a$.

Ex. 3d, An estate, which cost L. 8000, is let in tack

tack at the rent of L. 360 a-year; sought, at what number of years purchase it is bought, and what rate of interest the purchaser hath for his money?

Here $\frac{s}{a} = \frac{8000}{360} = 22.2\bar{2} = N$; and $\frac{a}{r} = \frac{360}{8000} = .045 = r = 4\frac{1}{2}$ per cent.

Of a Reversion in Perpetuity.

57. A reversion in perpetuity, or in fee-simple, being the same with the reversion of an annuity when the time prior to its commencement is equal to the duration of the annuity, its value may be found thus: Divide the value of the perpetuity, viz. $\frac{a}{r}$ by R^t , t being the time prior to its commencement, the quotient gives the reversion: Thus $\frac{a}{r R^t} = s$, the reversion in perpetuity. Hence the following

T A B L E.

Given,		
$a, t, R,$	$s = \frac{a}{r R^t}.$	Equ. 1.
$s, t, R,$	$a = r s \times R^t.$	2.
$s, t, a,$	$R^{t+1} - R^t = \frac{a}{r}.$	3.
$s, R, a,$	$t = \frac{1. a - L. r s}{L. R}.$	4.

Ex. 1st, Sought, the present value of a reversion in perpetuity, which is to commence 40 years hence, and whose yearly rent is L. 70, discounting interest at 4 per cent. Here,

$$\frac{a}{r R^t} = \frac{70}{.19204} = \text{L. } 364.5, \text{ the value sought.}$$

Ex.

Ex. 2d, There is a reversion in fee-simple fold for L. 500, which is to commence 25 years hence; fought, the yearly rent thereof, discounting interest at 5 per cent.

$$rs \times R^t = 25 \times 3.3863 = 84.6575, \text{ the rent fought.}$$

Ex. 3d, The reversion of an annuity of L. 20 a-year is fold for L. 222.735; fought, the duration of the annuity, reckoning interest at 5 per cent. Here,

$$\frac{L.a - L.rs}{L.R} = \frac{.2542716}{.0211893} = 12 \text{ years.}$$

Ex. 4th, There is an annuity of L. 35, to continue 19 years, whose reversion is fold for L. 256.9 ready money; fought, the rate of interest allowed the purchaser. Here,

$\frac{a}{r} = \frac{35}{.256.9} = .13624 = R^{20} - R^{19}$, found by Table II. betwixt 5 and 6 per cent.; therefore take two positions, &c.

1st Posit. $R = 1.05$ Result 277.652 Error + 20.744
 2d Posit. $R = 1.055$ Result 230.095 Error - 26.813
 By the question, Result 256.9 Diff. of Result of 1st & 2d = 47.557
 Then $47.557 : .005 :: 20.744 : .00218$ to be added to the 1st position, making the rate 5.218 per cent. nearly.

Ex. 5th, Which is most advantageous, a term of 16 years in an estate of L. 50 per annum; or the reversion of the same estate for ever, after the expiration of the said term, reckoning interest at 4 per cent.? Here,

By Equ. 1st, the Reversion = $13.3477 \times 50 = 667.385$
 By Table V. the Annuity = $11.11838 \times 50 = 555.919$

The Reversion is better than the Term of 16 years by L. 111.466

Ex.

Ex. 6th, A and B purchase an estate for L. 6700, which, at the term of Martinmas 1790, they agree to divide into two equal parts.

A sets his part for the yearly rent of L. 180, and has, besides, timber, and an old mansion-house, valued at L. 260.

B's part is under tack for 11 years to run, at the yearly rent of L. 143, and the timber upon it is valued at L. 22.

Sought, first, At what rate of interest the purchase was made; secondly, How much A's part is better than B's; thirdly, At what yearly rent must B set his part, 11 years hence, to make it equal to A's in its present value, reckoning interest at 5 per cent.

$$\frac{a}{s} = \frac{180}{3090} = .05825 = \text{L. } 5 : 16 : 6 \text{ per cent.}$$

$$\text{A's Part} = \frac{a}{r} = \frac{180}{.05} = 3600$$

$$\text{His timber and house} = 260$$

$$\text{A's value} = \text{L. } 3860$$

$$\text{B's Part} = \frac{a}{r} = \frac{143}{.05} = 2860$$

$$\text{His timber} = 22$$

$$\text{B's value} = \text{L. } 2882$$

So that A exceeds B in value L. 978. By 57. Equ. 2d, $\frac{978.00}{11.6935} = \text{L. } 83.636 = \text{B's rise of rent at the end of 11 years, to make his part equal in present value to that of A.}$

Remarks upon the Purchase of a Perpetuity.

58. 1st, A freehold-estate or perpetuity, purchased at the rate of interest r , is bought at a number of years purchase $= \frac{1}{r}$, and the rate of interest r , at which the purchase-money of a freehold estate is valued for N years purchase $= \frac{1}{N}$: So that N and r are reciprocal of, and discover one another. Thus:

O

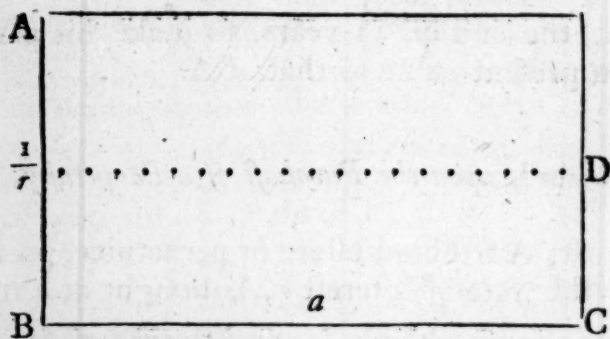
If

If $N=35$, then $r=.02857=2.857$ per cent.
 Or, If $r=.04$, then $N=25$ years purchase.

2d, N , the number of years purchase at which a perpetuity is bought or sold at a given rate of compound interest, is always equal to the number of years in which money doubles itself at the same rate of simple interest, seeing either of them $= \frac{1}{r}$, (23. & 56.) Thus :

When a purchase is made at	}	$2\frac{1}{2}$	per cent. then $N=$	40	years purchase,
		3		33.3	
		$3\frac{1}{2}$		28.57	
		4		25	
		$4\frac{1}{2}$		22.7	
		5		20	
		6		16.6	
		8		12.5, and vice versa,	

3d, The value of a perpetuity may be expressed by the area of a right-angled parallelogram AC,



whose breadth $AB = \frac{1}{r}$, and its length $BC = a$ the rent of the estate ; for $AB \times BC = \frac{a}{r} = s$, the value of the perpetuity ; and half of the parallelogram,

viz,

viz. $BD = \frac{a}{2r} =$ present value, or the reversion of an annuity a , whose duration is equal to the time in which a sum of money will double itself at the rate r of compound interest: For in Equation 4th, 47. in place of p , put its value in this case, viz. $\frac{a}{2r}$, then $t = \frac{L \cdot a - L \cdot \frac{a}{2}}{L \cdot R} = \frac{L \cdot \frac{a}{2}}{L \cdot R} =$ the time in which a sum of money will double itself at the rate r of compound interest.

4th, The method of buying and selling estates, at a certain number of years purchase, is very proper, being both simple, and well adapted to calculations upon the principles of compound interest, by which alone the value of a perpetuity can be ascertained.

5th, The purchaser of a perpetuity, if he can afford it, may venture to give, at least, 10 years purchase more for an estate than what would arise from the legal interest of 5 per cent. on account of the superior security, the chance of the rise of ground in its value; and because he is allowed compound interest upon the purchase-money, and other advantages.

A TABLE, Shewing the present Worth of L. 1 for Years.

Yrs.	3 per cent.	4 per cent.	5 per cent.	Yrs.	3 per cent.	4 per cent.	5 per cent.
51	.221463	.135301	.083051	71	.122619	.061749	.031301
52	.215013	.130097	.079096	72	.119047	.059374	.029811
53	.208750	.125093	.075330	73	.115580	.057091	.028391
54	.202670	.120282	.071743	74	.112214	.054895	.027039
55	.196767	.115656	.068326	75	.108945	.052784	.025752
56	.191036	.111207	.065073	76	.105772	.050754	.024525
57	.185472	.106930	.061974	77	.102691	.048801	.023357
58	.180070	.102817	.059023	78	.099700	.046924	.022245
59	.174825	.098963	.056212	79	.096796	.045120	.021186
60	.169733	.095060	.053536	80	.093977	.043384	.020177
61	.164789	.091404	.050986	81	.091240	.041716	.019216
62	.159990	.087889	.048558	82	.088582	.040111	.018301
63	.155330	.084508	.046246	83	.086002	.038569	.017430
64	.150806	.081258	.044044	84	.083497	.037085	.016600
65	.146413	.078133	.041946	85	.081065	.035659	.015809
66	.142149	.075128	.039949	86	.078704	.034287	.015056
67	.138009	.072238	.038047	87	.076412	.032968	.014339
68	.133989	.069460	.036235	88	.074186	.031700	.013657
69	.130086	.066788	.034509	89	.072027	.030481	.013006
70	.126297	.064219	.032866	90	.069928	.029309	.012387
				91	.067891	.028182	.011797

A TABLE Shewing the present Worth of L. 1 Annuity for Years.

Yrs.	3 per cent.	4 per cent.	5 per cent.	Yrs.	3 per cent.	4 per cent.	5 per cent.
51	25.9512	21.6174	18.3389	71	29.2460	23.4562	19.3739
52	26.1662	21.7475	18.4180	72	29.3650	23.5156	19.4037
53	26.3749	21.8726	18.4934	73	29.4806	23.5727	19.4321
54	26.5776	21.9929	18.5651	74	29.5928	23.6276	19.4592
55	26.7744	22.1086	18.6334	75	29.7018	23.6804	19.4849
56	26.9654	22.2198	18.6985	76	29.8076	23.7311	19.5094
57	27.1509	22.3267	18.7605	77	29.9102	23.7799	19.5328
58	27.3310	22.4295	18.8195	78	30.0099	23.8268	19.5550
59	27.5058	22.5284	18.8757	79	30.1067	23.8720	19.5762
60	27.6755	22.6234	18.9292	80	30.2007	23.9153	19.5964
61	27.8403	22.7148	18.9802	81	30.2920	23.9571	19.6156
62	28.0003	22.8027	19.0288	82	30.3805	23.9972	19.6339
63	28.1556	22.8872	19.0750	83	30.4665	24.0357	19.6514
64	28.3064	22.9685	19.1191	84	30.5500	24.0728	19.6680
65	28.4528	23.0466	19.1610	85	30.6311	24.1085	19.6838
66	28.5950	23.1218	19.2010	86	30.7098	24.1428	19.6988
67	28.7330	23.1940	19.2390	87	30.7862	24.1757	19.7132
68	28.8670	23.2635	19.2753	88	30.8604	24.2074	19.7268
69	28.9971	23.3302	19.3098	89	30.9324	24.2379	19.7398
70	29.1234	23.3945	19.3426	90	31.0024	24.2672	19.7522
			perpetuity		33.3333	25.0000	20.0000

A TABLE Shewing the Amount of L. 1 Annuity for Years.

Yrs.	3 per cent.	4 per cent.	5 per cent.	Yrs.	3 per cent.	4 per cent.	5 per cent.
51	117.181	159.774	220.815	71	238.512	379.862	618.955
52	121.696	167.165	232.856	72	246.667	396.057	650.903
53	126.347	174.851	245.499	73	255.067	412.899	684.448
54	131.138	182.845	258.774	74	263.719	430.415	719.670
55	136.072	191.159	272.713	75	272.631	448.631	756.654
56	141.154	199.806	287.348	76	281.810	467.577	795.486
57	146.388	208.798	302.716	77	291.264	487.280	836.261
58	151.780	218.150	318.851	78	301.002	507.771	879.074
59	157.333	227.876	335.794	79	311.032	529.082	924.027
60	163.053	237.991	353.584	80	321.363	551.245	971.229
61	168.945	248.510	372.263	81	332.004	574.295	1020.790
62	175.013	259.451	391.876	82	342.964	598.267	1072.830
63	181.264	270.829	412.470	83	354.253	623.197	1127.471
64	187.702	282.662	434.093	84	365.880	649.125	1184.845
65	194.333	294.968	456.789	85	377.857	676.090	1245.087
66	201.163	307.767	480.638	86	390.193	704.134	1308.341
67	208.198	321.078	505.670	87	402.898	733.299	1374.758
68	215.444	334.921	531.953	88	415.985	763.631	1444.496
69	222.907	349.318	559.551	89	429.465	795.176	1517.721
70	230.594	364.290	588.529	90	443.349	827.983	1594.607

These three Tables are continuations of Tab. III. IV. and V. of compound interest, and may be necessary, besides other purposes, in the solution of questions in annuities on lives, when n , the complement of life, exceeds 50.

CHAP.

C H A P. VIII.

Of ANNUITIES on LIVES.

59. WAS human life certain of continuing to a given period, the sum of the series $\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} (n) = \frac{1 - \frac{1}{R^n}}{r}$ (47.) would express, at a given rate, the value of a person's life who had yet n years to live; but seeing there is much contingency in the case, as we find by daily experience, Tables of the Probabilities of Life, called also Tables of Observations, have been constructed from the bills of mortality; from whence the value of life, at a given age and rate of interest, might be computed. Of these take the following Table, constructed by Dr Halley from the bills of mortality at Breslaw, as an example.

TABLE

TABLE I. Shewing the Probabilities of Life at Breslaw, Constructed by Dr Halley from the Bills of Mortality in that City.

Age.	Living.	Dead.	Age.	Living.	Dead.	Age.	Living.	Dead.
0	1138	238	31	523	8	62	222	10
1	1000	145	32	515	8	63	212	10
2	855	57	33	507	8	64	202	10
3	798	38	34	499	9	65	192	10
4	760	28	35	490	9	66	182	10
5	732	22	36	481	9	67	172	10
6	710	18	37	472	9	68	162	10
7	692	12	38	463	9	69	152	10
8	680	10	39	454	9	70	142	11
9	670	9	40	445	9	71	131	11
10	661	8	41	436	9	72	120	11
11	653	7	42	427	10	73	109	11
12	646	6	43	417	10	74	98	10
13	640	6	44	407	10	75	88	10
14	634	6	45	397	10	76	78	10
15	628	6	46	387	10	77	68	10
16	622	6	47	377	10	78	58	9
17	616	6	48	367	10	79	49	8
18	610	6	49	357	11	80	41	7
19	604	6	50	346	11	81	34	6
20	598	6	51	335	11	82	28	5
21	592	6	52	324	11	83	23	4
22	586	7	53	313	11	84	19	4
23	579	6	54	302	10	85	15	4
24	573	6	55	292	10	86	11	3
25	567	7	56	282	10	87	8	3
26	560	7	57	272	10	88	5	2
27	553	7	58	262	10	89	3	2
28	546	7	59	252	10	90	1	1
29	539	8	60	242	10	91	0	0
30	531	8	61	232	10			

Such a table as this of the probabilities of life, is formed on the principle, "that the number dying annually, after every particular age, is equal to the number living at that age," and, of

of consequence, is possessed of the following properties:

1st, The several numbers in column 2. or 3. from infancy to the utmost extent of life, will represent the persons living, or who die yearly, at their corresponding ages.

2^d, The sum of all the persons alive or dead in column 2. or 3. will express the number of inhabitants, or the number of persons who die yearly, and these last, it is manifest, will be equal to 1238 who are born in any one year; also the sum of the persons dying yearly, from any one age and upwards, will be equal to the number living at that age.

3^d, As the 1000 children, 1 year old, were not all born at the beginning of the year, but successively from day to day, they, as well as the persons living at any succeeding age, are represented in the table half a year older, at a medium, than they are in fact; of consequence, in questions respecting the expectation of a single life, (See Art. 4.) half a year ought to be deducted from the answers given by the Table. Hence may be derived the following Articles:

Art. 1. To find the chance or probability which a person of a given age hath of living a proposed number of years. This is discovered by the proportion which the living, at the proposed age, have to those who have died since the given one. Thus, Supposing a person of 30 years of age, then his chance of living is to that of dying,

As $523 : 8 = 65 : 1$, in 1 year.
and $472 : 59 = 8 : 1$, in 7 years.

Hence,

Hence, to find how long a person of a given age hath the chance of living, take half the number of persons living at the given age, and opposite to it in col. 1st, you have the years required.

Thus, $\frac{531}{2} = 265$, which is found in the table between 57 and 58 years; so that a person aged 30 hath an equal chance of living between 27 and 28 years longer.

Art. 2. To find the probability of life's continuing, or of its failing in 1, 2, 3, &c. years, in a person of a given age, when the certainty of the present moment, or of death, is expressed by unity. Here, supposing a person of 30 years of age, out of 531 chances, he hath 523 of living one year, and eight chances of dying in that time. But by the doctrine of chances, the probability for the happening of an event is expressed by a fraction, whose numerator is the number of chances for its happening, and denominator, the sum of the chances both of its happening and failing. Hence,

$$\left. \begin{array}{l} \frac{523}{531} \\ \frac{8}{531} \end{array} \right\} = \text{Probability that such a person} \left\{ \begin{array}{l} \text{will live 1 year.} \\ \text{will die in that time.} \end{array} \right.$$

But the sum of these fractions is equal to unity; therefore, having one, we can find the other by subtraction. Hence, $\frac{523}{531}$, $\frac{515}{531}$, $\frac{507}{531}$, will express the probabilities that such a person will live 1, 2, 3 years longer; and $\frac{8}{531}$, $\frac{16}{531}$, $\frac{24}{531}$, those of his dying in that time.

Art. 3d. Out of a certain number of persons of a given age, to find how many of them have a chance of dying yearly.

As the tabular number, at the given age, is to those who die out of them, so is the number of persons alive, to those who die yearly.

Ex. Out of 135 persons, whose age, at a medium, is 55 years, it is required, to find how many have a chance of dying yearly, till the last is extinct at the age of 91 years.

$292 : 10 :: 135 : 4.623$ who die the 1st year;
and $282 : 10 :: 130.377 : 4.623$ 2d year, &c.

Or rather thus: As these persons are in life from 30 years of age and upwards, nearly in the same proportion as the tabular numbers from the same age; hence,

$15031 : 531 :: 135 : 4.77$ who die the 1st year;
and $14500 : 523 :: 130.23 : 4.70$ 2d year,
&c.

Art. 4th. To find the expectation of life, that is, the number of years which mankind enjoy, taken one with another, either from birth or any age proposed.

Divide the number of persons alive at the given age and upwards, by the number alive at the given age, the quotient, minus .5, gives the answer. Thus,

$\frac{35173}{1238}$ (minus .5) = 28, the expectation of an infant.

Or thus: The expectation of life being equal to the sum of the probabilities that a person shall live from the present instant, 1, 2, 3, &c. years; the expectation

expectation of life in a person, for instance, of 30 years of age, is expressed by the sum of the fractions, viz.

$\frac{531}{531} + \frac{523}{531} + \frac{515}{531}$, &c. carried on to the utmost extent of life, (minus .5) = 27.87.

Art. 5th. To calculate the value of L. 1 annuity upon a life of a given age, and at a given rate of interest.

By Art. 2d. the probabilities of the continuance of life in a person, for instance, of 30 years of age, for 1, 2, 3, &c. years, are expressed by a series of fractions, viz.

$\frac{523}{531}, \frac{515}{531}, \frac{507}{531}$, &c. But independent of chance, the terms of the series, viz.

$\frac{1}{R}, \frac{1}{R^2}, \frac{1}{R^3}, \dots, \frac{1}{R^n}$, shew the present value of L. 1 due at the end of 1, 2, 3, $\dots, n$, years. Therefore, combining these two series together,

$\frac{523}{531R} + \frac{515}{531R^2} + \frac{507}{531R^3}$, &c. carried on to the utmost extent of life, will express the value of L. 1 annuity, upon the life of a person aged 30 years, at a given rate of interest. But as the calculation of 60 terms of this series must prove a work of labour, let $N=1.44$ denote at 4 per cent. the value of a life aged 86, which is easily found by the above method, and $\frac{a}{b} = \frac{11}{15}$, the probability that a life, aged 85, will survive 1 year; then,

$\frac{a}{bR} \times \overline{1+N} = 1.72$, is the value of a life aged 85 years. Proceeding thus, step by step, the values
P 2 of

of lives, by any table of observations, may be found.

Art. 6th. To find the amount of the population in any city or county in the kingdom, from their annual births or burials.

If the annual average of births and burials be equal, either of them multiplied by the expectation of an infant's life taken from a table adapted to that district, will produce the amount of the population thereof. If this annual average be unequal, which is most likely; in this case, as the population is as much increased by the superior number of births, as it is diminished by the comparative defect in the number of burials; therefore, the half of their sum, multiplied by an infant's expectation of life, call it 40, (see Table VIII.) will produce the number of inhabitants.

Ex. By the collector's accounts of the duty upon baptisms, &c. within the bounds of the synod of Fife, anno 1790,

The annual average of baptisms = 2484
 of burials = 1536

Half their sum = 2010
 Multiplied by 40

Produces 80400, the number of inhabitants within the bounds of the synod of Fife.

60. *Of the Hypothesis of Equal Decrements.*

Mr de Moivre hath invented an hypothesis, whereby the values of annuities on lives may be determined

determined nearly as accurately as if they had been deduced year by year from a table of observations; namely, "That the probabilities of life, after a certain age, decrease in arithmetical progression." Thus, supposing, for instance, 36 persons, each of the age of 50, if, after one year expired, there remained but 35, after two 34, after three 33, &c. it is evident that such lives would be extinct in 36 years; and that the probabilities of living 1, 2, 3, &c. years from the age of 50, would be expressed by the fractions,

$\frac{35}{36}, \frac{34}{36}, \frac{33}{36},$ &c, which decrease in arithmetical progression: And supposing further, that the age of 86 is the utmost probable extent of human life; the complement of life of a given age will be the number of years which that age wants of 86. Thus, 36 will be the complement of a life aged 50 years.

Admitting, then, that the decrements of life, after 10 years of age, are nearly equal; the following Problems will more easily be resolved, than by a table of observations.

P R O B L E M I.

61. Supposing the decrements of life to be equal, it is required, to find the value of L. 1 annuity upon a life of a given age, and at a given rate of compound interest.

Let n represent the complement of life.

$p = \frac{1}{R^n}$ = the value of L. 1, due at the end of n years.

$P = \frac{1-p}{r}$ = of L. 1 annuity for n years.

$N =$

N = the value of L. 1 annuity upon a life of a given age. Now, since the probabilities of life decrease in arithmetical progression, they may be represented by the terms of the following series, viz.

$\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n} \dots \frac{n-n}{n}$, which expressions may also be considered as the values of each payment of the annuity in the parts of L. 1. But the terms of the series, $\frac{1}{R}, \frac{1}{R^2}, \frac{1}{R^3} \dots \frac{1}{R^n}$, shew the present values of L. 1, due at the end of 1, 2, 3, $\dots n$ years; therefore combining these two series together,

$\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3} (n) = N$, will express the value of an annuity upon a life, whose complement is n , at a given rate of interest.

Which series may be divided into two other series, viz.

$$\frac{n}{n} \times \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} (n) = \frac{1}{1} \times \frac{1-p}{r}; \quad (47) \quad \text{and} \quad -\frac{1}{n} \times \frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3} (n) = \frac{1}{n} \times \frac{RP}{r} - \frac{np}{r} *.$$

Hence,

* Sought, the value of n terms of the series, $\frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3}$, &c.

If the terms of the series $\frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3}$, &c. be multiplied

$$\text{by } \overline{R-1}^2 = R^2 + 2R + 1$$

The product $= R + 0 + 0$. Hence, $\frac{R}{R-1}^2 = \frac{R}{r^2}$

is the value of the series indefinitely continued.

Now

Hence, $\frac{1-p}{r} - \frac{RP}{nr} + \frac{p}{r} = N$; but $\frac{1-p}{r} + \frac{p}{r} = \frac{1}{r}$.

Hence, $\frac{1}{r} - \frac{RP}{nr} = \frac{1 - \frac{R}{n}P}{r}$ } = N , the value
Or, $P - \frac{RP - np}{nr}$ } sought.

Ex. It is required, to find the present value of L. 1 annuity upon the life of a person aged 36 years, reckoning interest at 4 per cent. Here,

$n=50$, $P=21.482$, $R=1.04$, and $r=.04$; hence,

$$\frac{1 - \frac{R}{n}P}{r} = \frac{.5532}{.04} = 13.83, \text{ the value sought.}$$

PROBLEM II.

Supposing the decrements of life to be equal, it is required to find the present value of L. 1 an-

Now the terms, which follow the n term, in the given series,

$$\text{are } \frac{n+1}{R^{n+1}} + \frac{n+2}{R^{n+2}} + \frac{n+3}{R^{n+3}}, \text{ \&c. ad infinitum.}$$

$$\text{Or, } \frac{1}{R^n} \times \frac{n+1}{R} + \frac{n+2}{R^2} + \frac{n+3}{R^3}, \text{ \&c. ad infinitum.}$$

Which series may be divided into two other series, viz.

$$\frac{1}{R^n} \times \frac{n}{R} + \frac{n}{R^2} + \frac{n}{R^3}, \text{ \&c. } = \frac{n}{R^n} \times \frac{1}{r} (40) = \frac{np}{r}; \text{ and,}$$

$$\frac{1}{R^n} \times \frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3}, \text{ \&c. } = \frac{1}{R^n} \times \frac{R}{r^2} = \frac{Rp}{r^2}, \text{ by the above. Hence,}$$

if the sum of these two series be subtracted from the former, there will remain

$$\frac{R}{r^2} - \frac{Rp}{r^2} - \frac{np}{r}, \text{ or, } \frac{1-p}{r^2} \times R - \frac{np}{r} = \frac{RP}{r} - \frac{np}{r} = \text{the value of}$$

the series $\frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3}, (n).$

nunity,

nunity, to continue during the joint lives of two persons, whose respective complements of life are n and m , (m being the greater number) at a given rate of compound interest.

62. Since the probability of the continuance of the life, whose complement is n , for one year, is $\frac{n-1}{n}$, and that of the life whose complement is m , $\frac{m-1}{m}$; and since these two events are independent of each other, it follows, that the terms of the series, $\frac{n-1}{n} \times \frac{m-1}{m}$, $\frac{n-2}{n} \times \frac{m-2}{m}$, $\frac{n-3}{n} \times \frac{m-3}{m}$, &c. will respectively represent the probabilities of their continuing together, 1, 2, 3, &c. years; which probabilities may be considered as the values of the several payments of the annuity, in the parts of L. 1, which will become due at the end of 1, 2, 3, &c. years; and the value of L. 1 annuity upon the two joint lives, will be expressed by the following series.

$$\frac{n-1 \times m-1}{nmR} + \frac{n-2 \times m-2}{nmR^2} + \frac{n-3 \times m-3}{nmR^3} \text{ \&c.}$$

of which series n terms only will be useful, n being the complement of the oldest life. Now if the numerators of the fractions in the above series be expanded by multiplication, they will appear in this form.

$$\frac{n-1 \times m-1}{nm} = nm - m + n \times 1 + 1.$$

$$\frac{n-2 \times m-2}{nm} = nm - m + n \times 2 + 4.$$

$$\frac{n-3 \times m-3}{nm} = nm - m + n \times 3 + 9, \text{ \&c.}$$

Hence, the above series may be divided into three other series,

viz.

$$\text{viz. } \frac{nm}{nm} \times \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}, (n) = \frac{nm}{nm} \times \frac{1-p}{r} \quad (47.)$$

$$- \frac{m+n}{nm} \times \frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3}, (n) = \frac{m+n}{nm} \times \frac{1-pR \times \frac{np}{r^2} - \frac{np}{r}}{r^2}$$

(By note, Prob. 1.)

$$+ \frac{1}{nm} \times \frac{1}{R} + \frac{4}{R^2} + \frac{9}{R^3}, (n) = \frac{1}{nm} \times \frac{1-p \times R + 1 \times R - 2nRp}{r^3} - \frac{n^2p}{r^2} *$$

The sum of which three series, being properly ranged, may stand thus,

$$\begin{aligned} & \frac{nm}{nm} \times \frac{1-p}{r} \\ & + \frac{m+n}{nm} \times \frac{np}{r} - \frac{m+n}{nm} \times \frac{1+p \times R}{r^2} \\ & - \frac{1}{nm} \times \frac{np}{r} - \frac{1}{nm} \times \frac{2npR}{r^2} + \frac{1}{nm} \times \frac{1-p \times R + 1 \times R}{r^3} \end{aligned}$$

$$\text{Their sum} = \frac{1}{r} - \frac{m+n \times R - m - n \times Rp}{nm \times r^2} + \frac{1-p \times R + 1 \times R}{nm \times r^3}$$

= the value of the annuity.

Ex.

* If the series $\frac{1}{R} + \frac{4}{R^2} + \frac{9}{R^3}$ &c. be multiplied

by $R-1^3 = R^3+3R^2+3R-1$

the product $= R^3 + R + 0 = R+1 \times R$; therefore,

$\frac{R+1 \times R}{r^3}$ = the value of the above series indefinitely continued.

Now the terms which follow the n term in the series are,

$$\frac{n+1^2}{R^{n+1}} + \frac{n+2^2}{R^{n+2}} + \frac{n+3^2}{R^{n+3}}, \text{ \&c.}$$

$$\text{Or, } \frac{nn+2n+1}{R^{n+1}} + \frac{nn+4n+4}{R^{n+2}} + \frac{nn+6n+9}{R^{n+3}}, \text{ \&c.}$$

Which series may be divided into three others, viz,

$$\frac{nn}{R^n} \times \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}, \text{ \&c.} = \frac{nn}{R^n} \times \frac{1}{r} \quad (40.)$$

$$\frac{2n}{R^n} \times \frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3}, \text{ \&c.} = \frac{2n}{R^n} \times \frac{R}{r^2} \quad (\text{last note})$$

$$\frac{1}{R^n} \times \frac{1}{R} + \frac{4}{R^2} + \frac{9}{R^3}, \text{ \&c.} = \frac{1}{R^n} \times \frac{n+1 \times R}{r^3}, \text{ as above.}$$

Q

Ia

Ex. It is required to find the present value of L. 1 annuity, to continue during the joint lives of two persons, whose respective ages are 54 and 43 years, reckoning interest at 4 per cent. Here $n=32$, $m=43$, $p=.285$, $R=1.04$ and $r=.04$. Hence,

$$\begin{aligned} & \frac{1}{r} - \dots - \dots = 25.000 \\ & + \frac{1-p \times R+1 \times R}{nm \times r^3} = \frac{1.5168}{.088} = \frac{17.224}{42.224} \\ & - \frac{m+n \times R - m-n \times R p}{nm \times r^2} = \frac{74.739}{2.2} = \frac{33.947}{} \\ & \text{The value of the annuity} = 8.277 \end{aligned}$$

When the single lives are equal, $m+n=2n$, $\overline{m-n} \times R p = 0$, $nm=n^2$, and the negative quantity $= -\frac{2R}{nr^2}$

Ex. Sought, the value of L. 1 annuity to continue during the joint continuance of two lives, each aged 48 years, reckoning interest at 4 per cent. Here,

$$\begin{aligned} \frac{1}{r} + \frac{1-p \times R+1 \times R}{n^2 \times r^3} &= 42.785. \\ - \frac{2R}{nr^2} &= 34.215. \end{aligned}$$

The value sought = 8.570.

In which three series, making $p = \frac{1}{R^n}$, and subtracting their sum from the above found value of the whole series, there will remain $\frac{1-p \times R+1 \times R}{r^4} - \frac{2npR}{r^2} - \frac{np}{r}$ = the value of n terms thereof.

P R O B-

PROBLEM III.

To find the present value of L. 1 annuity, either upon the joint continuance, or during the longest liver, of any number of lives, at a given rate of interest.

63. 1st, As the value of joint lives enters an ingredient into almost every question respecting life-annuities, this value, at 4 per cent. may be discovered by the following concise and accurate method. Let n , m , p , and t , represent the complements of life; c , the arithmetical mean complement of all the lives, excepting the oldest, whose value, at 4 per cent. is N ; and $N - \frac{a}{b}N$, the value of the joint lives. Then,

In the case of 2, 3, 4, equal joint lives, with a small correction, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{2}$, = the *ratio* of the value of the joint lives to that of the single one; hence,

$\frac{a}{b}$, with a small correction, $= \frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2} = 1$ — the *ratio* of the value of the joint lives to that of the single one.

Again, in the case of 2 } equal joint lives, $n = \begin{cases} m. \\ \text{— of 3 or 4} \end{cases} c.$

But when the lives are unequal, $\frac{n}{m}$ or $\frac{n}{c}$ will correct the irregularity arising from the inequality of the lives. Hence the following

T A B L E.

Lives.	Corrections at 4 per cent.								
	$n = 76$	66	56	51	46	36	26	16	6
2	$N - \frac{n-12}{4^m} N$	-7	-3	-1	+1	+3	+4	+5	+3
3	$N - \frac{n+8}{3^c} N$	+9	+10	+10	+10	+10	+9	+6	+3
4	$N - \frac{n-6}{2^c} N$	-3	-1	0	+1	+2	+3	+2	+1

Ex. Sought, the value of two joint lives, whose respective ages are 46 and 30 years, at 4 per cent. Here $n = 40$, $m = 56$, and $N - \frac{n+2.2}{4^m} N = 12.135 - 2.286 = 9.849$, the value.

Or, the present value of joint lives, at a given rate of interest, may be found universally. Thus:

Let a represent the expectation of their joint continuance, (see supplement to problem 18.); b , that of the oldest; and A , the number of years, by Table V. of annuities certain, corresponding to N , the value of the oldest life*. Then, seeing $b : a :: A : \frac{a}{b} A$, the value by the said table, answering to $\frac{a}{b} A$ years certain †, (or in the case of two joint lives, the value corresponding to $A - \frac{n}{3^m} A$ years certain), will be that of their joint continuance.

Ex. Let there be five equal lives, each aged 32

* Let the age be 32, whose value, at 4 per cent is 14.411; the next less, to which, by Table V. is 14.029, answering to 21 years; their difference is .382, and the increase of value for one year is .422; hence, $\frac{.382}{.422} = .905$, to be annexed to 21, making, in all, 21.905 years.

† Let the number of years certain be 32.86, the value of 32 years, at 4 per cent. is 17.873, and the increase of value for one year is .274; hence $.274 \times .86 = .235$ to be added to 17.873, making their sum = 18.108, the value sought.

years;

years; it is required to find the value of their joint continuance, at 4 per cent. Here, $a=9$, $b=27$, and $A=21.9$; hence, $\frac{a}{b}A=7.3$ years, whose value, viz. 6.46, is that of the joint continuance of these five lives.

Or, the present value of L. 1 annuity upon the joint continuance of any number of lives, at a given rate, may be found universally, thus:

2d, Let N =expectation of the oldest life, whose complement is n ; x =expectation of the joint continuance of all the lives; and A =number of years certain, corresponding to the value of the oldest life. Then,

Seeing $N : x :: A : \frac{x}{N}A$, the value corresponding to $\frac{x}{N}A$ years certain, (or in the case of two lives, the value answering to $A - \frac{n}{3^m}A$ years), will be that of their joint continuance.

Ex. Let there be three lives, whose respective ages are 46, 36, and 26 years; it is required to find the value of their joint continuance, reckoning interest at 5 per cent. Here,

$N=20$, $x=12$, and $A=16.35$; hence,

$\frac{12}{20} \times 16.35 = 9.81$ years, whose value, viz. 7.6, is that of the joint continuance of these three lives.

In the case of m equal joint lives, $\frac{x}{N} = \frac{2}{m+1}$.

23d, To find the value of L. 1 annuity upon the longest liver of any number of lives, at a given rate.

Let n =complement of the mean life, whose expectation is N ; X or x =expectation of the longest liver, or of joint continuance of all the lives; and $d=N$ —the number of years certain, corresponding

ing to the value of the mean life, at a given rate. Then,

Seeing $n - N = N : n - X = x :: d : \frac{x}{N} \times d$, the value corresponding to $X - \frac{x}{N} \times d$ years certain, will be that of the longest liver.

Ex. Let there be four equal lives, whose common age is 36 years; sought, the value of the longest liver, reckoning interest at 4 per cent. Here,

$n = 50$, $N = 25$, $X = 40$, $x = 10$, and $d = 4.46$. Hence,

$40 - \frac{4.46}{25} \times 10 = 38.2$ years, whose value, viz. 19.4, is that of the longest liver of these 4 lives.

In the case of m equal lives, $X - \frac{2d}{m+1}$ = the number of years certain, corresponding to the value of the longest liver.

These two methods of finding the value of the joint continuance, &c. of any number of lives, are more accurate than Mr de Moivre's Theorms for the same purpose; and more concise than those of Mr Dodson in his Mathematical Repository, vol. 2. and may be applied in finding the value of lives, by a table of observations, as well as by the hypothesis.

Ex. It is required to find, by Table VIII. and IX. the value of the joint continuance of two lives, whose respective ages are 30 and 25 years, at 4 per cent. Here,

$n = 65.08$, $m = 69.98$, and $A = 27.244$; hence,

$A - \frac{n}{3m} A = 18.8$ years, whose value, viz. 13, is that of the joint continuance of these two lives.

TABLE

TABLE II. Shewing the present Values of L. 1
Annuity on a single Life, according to Mr de
Moivre's Hypothesis of equal Decrements.

Age.	3 per cent.	4 per cent.	5 per cent.	Age.	3 per cent.	4 per cent.	5 per cent.
9-10	19.868	16.882	14.607	48	13.012	11.748	10.679
8-11	19.736	16.791	14.544	49	12.764	11.548	10.515
7-12	19.604	16.698	14.480	50	12.511	11.344	10.348
13	19.469	16.604	14.412	51	12.255	11.135	10.176
6-14	19.331	16.508	14.342	52	11.994	10.921	9.999
15	19.192	16.410	14.271	53	11.729	10.702	9.817
16	19.050	16.311	14.197	54	11.457	10.478	9.630
5-17	18.905	16.209	14.125	55	11.183	10.248	9.437
18	18.759	16.105	14.047	56	10.902	10.014	9.239
19	18.610	15.999	13.970	57	10.616	9.773	9.036
4-20	18.458	15.891	13.891	58	10.325	9.527	8.826
21	18.305	15.781	13.810	59	10.029	9.275	8.611
22	18.148	15.669	13.727	60	9.727	9.017	8.389
23	17.990	15.554	13.642	61	9.419	8.753	8.161
3-24	17.827	15.437	13.555	62	9.107	8.482	7.926
25	17.664	15.318	13.466	63	8.787	8.205	7.684
26	17.497	15.197	13.375	64	8.462	7.921	7.435
27	17.327	15.073	13.282	65	8.132	7.631	7.179
28	17.154	14.946	13.186	66	7.794	7.333	6.915
29	16.979	14.816	13.088	67	7.450	7.027	6.643
30	16.800	14.684	12.988	68	7.099	6.714	6.362
2-31	16.620	14.549	12.885	69	6.743	6.394	6.073
32	16.436	14.411	12.780	70	6.378	6.065	5.775
33	16.248	14.270	12.673	71	6.008	5.728	5.468
34	16.057	14.126	12.562	72	5.631	5.383	5.152
35	15.867	13.979	12.449	73	5.246	5.029	4.826
36	15.666	13.829	12.333	74	4.854	4.666	4.489
37	15.465	13.676	12.214	75	4.453	4.293	4.143
38	15.260	13.519	12.091	76	4.046	3.912	3.784
39	15.053	13.359	11.966	77	3.632	3.520	3.415
40	14.842	13.196	11.837	78	3.207	3.111	3.034
41	14.626	13.028	11.705	79	2.776	2.707	2.641
42	14.407	12.858	11.570	80	2.334	2.284	2.235
43	14.185	12.683	11.431	81	1.886	1.850	1.816
44	13.958	12.504	11.288	82	1.429	1.406	1.384
45	13.728	12.322	11.142	83	0.961	0.950	0.937
46	13.493	12.135	10.992	84	0.484	0.481	0.476
47	13.254	11.944	10.837	85	0.000	0.000	0.000

Table II. is constructed by the theorem $\frac{1-\frac{R}{n}p}{r}$, which expresses the value of a single life, whose complement is n .

Or thus, let the sum of the following series, viz.

$$\frac{1}{R}.$$

$$\frac{1}{R} + \frac{1}{R^2}$$

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}$$

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \frac{1}{R^4}$$

&c.

Their sum = $\frac{n-1}{n} \frac{1}{R} + \frac{n-2}{n} \frac{1}{R^2} + \frac{n-3}{n} \frac{1}{R^3} \dots + \frac{1}{R^{n-1}}$ = the

present worth of L. 1 annuity upon a single life, whose complement is n , at a given rate of interest, and is the same with the series of equal decrements, Problem I. wanting the last term, viz. $\frac{n-n}{n} R^n = 0$. Hence the sum of $n-1$ terms, in Table V. of annuities certain, divided by n , will give the value of a life whose complement is n .

Ex. Let the age be 70, whose value is sought, at 4 per cent. Here, $n=16$, and

15 terms of Table V. = 97.04
divided by 16 = 6.065 $\left\{ \begin{array}{l} \text{the value} \\ \text{sought.} \end{array} \right.$

This affords an accurate and easy method of finding the value of single or conjoint lives, &c. at a given rate, not only by the hypothesis, but also by

by a table of observations, when the expectation of life is known.

Ex. By Table VIII. $22.42 =$ expectation of the joint continuance of two lives, whose respective ages are 30 and 25; sought, their value at 4 per cent. Here, $n = 44.84$ and 43.84 terms of Table V. $= \frac{582.96}{44.84} = 13$, { the value of the joint continuance of these two lives.

Again, let N denote the value of a single life of a given age, $\frac{a}{b} = \frac{n}{n+1}$ the probability that a life will continue for one year preceding that corresponding to N ; then,

$\frac{a}{bR} \times 1 + N =$ the value of a single life one year younger than that corresponding to N .

As the difference of the terms in Table V. of annuities certain doth constantly decrease, and seeing $n-1$ terms are divided by n , in finding the value of a life whose complement is n ; hence, the present value of a single life will always be less than that of its expectation, considered as so many years certain.

Whereas the amount of an annuity on a given life (see Table IV.) is greater than that during its expectation; yet such is the accuracy of calculation, that the sum of $N \times n$, improved during life, will exactly balance the n number of life-annuities, each of whose values is N , whether they be paid regularly as they become due, or are discharged by single payments at the close of each life.

R

TABLE

TABLE III. Shewing the Value of L. 1 Annuity
on the Joint Continuance of two Lives, sup-
posing the Decrements of Life to be equal.

Age	Age. 3 pr cent.	4 pr cent.	5 pr cent.	Age.	Age. 3 pr cent.	4 pr cent.	5 pr cent.			
10	10	15.206	13.342	11.855	30	30	12.434	11.182	10.133	
	15	14.878	13.093	11.661		35	12.019	10.838	9.854	
	20	14.593	12.808	11.430		40	11.502	10.428	9.514	
	25	14.074	12.480	11.182		45	10.898	9.936	9.112	
	30	13.585	12.102	10.884		50	10.183	9.345	8.620	
	35	13.025	11.665	10.537		55	9.338	8.634	8.018	
	40	12.381	11.156	10.128		60	8.338	7.779	7.280	
	45	11.644	10.564	9.646		65	7.161	6.748	6.373	
	50	10.796	9.871	9.074		70	5.777	5.595	5.254	
	55	9.822	9.059	8.391		35	35	11.632	10.530	9.600
60	8.704	8.105	7.572	40	11.175		10.157	9.291		
65	7.417	6.980	6.585	45	10.622		9.702	8.913		
70	5.936	5.652	5.391	50	9.955		9.149	8.450		
15	15	14.574	12.860	11.478	55		9.156	8.476	7.879	
	20	14.225	12.593	11.266	60		8.202	7.658	7.172	
	25	13.822	12.281	11.022	65		7.066	6.662	6.294	
	30	13.359	11.921	10.736	70		5.718	5.450	5.203	
	35	12.824	11.501	10.402	40		40	10.777	9.826	9.014
	40	12.207	11.013	10.008			45	10.283	9.418	8.671
	45	11.496	10.440	9.541		50	9.677	8.911	8.244	
	50	10.675	9.767	8.985		55	8.936	8.283	7.710	
	55	9.727	8.975	8.318		60	8.038	7.510	7.039	
	60	8.632	8.041	7.515		65	6.951	6.556	6.198	
65	7.377	6.934	6.544	70		5.646	5.383	5.141		
70	5.932	5.623	5.364	45		45	9.863	9.063	8.370	
20	20	13.904	12.341			11.067	50	9.331	8.619	7.987
	25	13.531	12.051			10.840	55	8.662	8.044	7.500
	30	13.098	11.711		10.565	60	7.831	7.332	6.875	
	35	12.594	11.314		10.278	65	6.807	6.425	6.080	
	40	12.008	10.847		9.870	70	5.556	5.300	5.063	
	45	11.325	10.297	9.420	50	50	8.892	8.235	7.660	
	50	10.536	9.648	8.880		55	8.312	7.738	7.230	
	55	9.617	8.879	8.233		60	7.568	7.091	6.664	
	60	8.549	7.967	7.448		65	6.623	6.258	5.926	
	65	7.308	6.882	6.495		70	5.442	5.193	4.964	
70	5.868	5.590	5.333	55		55	7.849	7.332	6.873	
25	25	13.192	11.786		10.621	60	7.220	6.781	6.386	
	30	12.792	11.468		10.367	65	6.379	6.036	5.724	
	35	12.333	11.095		10.067	70	5.291	5.053	4.833	
	40	11.776	10.655		9.708	60	60	6.737	6.351	6.001
	45	11.130	10.131		9.278		65	6.043	5.730	5.444
	50	10.374	9.509	8.761	70		5.081	4.858	4.653	
	55	9.488	8.766	8.134	65		65	5.547	5.277	5.031
	60	8.452	7.880	7.371			70	4.773	4.571	4.385
	65	7.241	6.826	6.440			70	70	4.270	4.104
	70	5.826	5.551	5.294						

In the construction of Table III. Dr Price made use of the theorem $V = \frac{v+1}{m} \times m - n - 2v - 1 \times \frac{2}{n} + 2v = NM$, expressing the value of the joint continuance of two lives, whose complements are n and m . Here,

v represents $\begin{cases} \text{the perpetuity} = \frac{1}{r}, \text{ at a given rate.} \\ \text{the value of L. 1. annuity for } n \text{ years certain.} \end{cases}$

If only one, or neither of the ages, are to be found in the table, the value of their joint continuance, at a given rate, may be found by the theorem, $N - \frac{n+1}{4m} N$.

Ex. Let there be two lives, whose respective ages are 38 and 22 years; sought, the value of their joint continuance, at 4 per cent. Here $N - \frac{n+1}{4m} N = 13.519 - 2.55 = 10.969$, the value sought.

Or thus,

1st, Let there be two lives, aged 35 and 22 years; sought, the value of their joint continuance, at 4 per cent.

From the value of 20 by 35 = 11.314

Subtract that of 25 by 35 = 11.095

The remainder $\times \frac{2}{5} = .4 \times .219 = .088$; which product, subtracted from 11.314, leaves 11.226 = the value sought.

2d, Let there be two lives, aged 38 and 22 years; sought, the value of their joint continuance, at 4 per cent.

R 2

From

From the value found above = 11.226

Subtract that of 22 by 40 = 10.770

The remainder $\times \frac{3}{5} = .6 \times .456 = .274$, which product, subtracted from 11.226, leaves 10.952, the value sought.

TABLE IV. Shewing the amount of L. 1 Annuity foreborn, during a Life of a given Age, at 4 per cent.

Ages.	Amount.	Ages.	Amount.	Ages.	Amount.	Ages.	Amount.
9-10	128.81	29	66.58	48	31.51	67	11.36
8-11	124.54	30	64.20	49	30.15	68	10.59
7-12	120.41	31	61.89	50	28.89	69	9.84
13	116.40	32	59.65	51	27.61	70	9.10
6-14	112.52	33	57.48	52	26.37	71	8.38
15	108.74	34	55.37	53	25.16	72	7.67
16	105.11	35	53.32	54	24.00	73	6.98
5-17	101.66	36	51.34	55	22.85	74	6.31
18	98.13	37	49.41	56	21.75	75	5.65
19	94.81	38	47.53	57	20.67	76	5.02
4-20	91.58	39	45.72	58	19.62	77	4.39
21	88.44	40	43.95	59	18.61	78	3.79
22	85.41	41	42.24	60	17.62	79	3.20
23	82.47	42	40.56	61	16.66	80	2.63
3-24	79.61	43	38.96	62	15.72	81	2.08
25	76.85	44	37.39	63	14.80	82	1.54
26	74.17	45	35.87	64	13.92	83	1.01
27	71.56	46	34.38	65	13.06	84	0.50
28	69.04	47	32.92	66	12.25	85	0.00

In the construction of this table, which hath not been attempted by any author,

Let the sum of the following series, viz.

$$\begin{aligned}
 &1 + R \\
 &1 + R + R^2 \\
 &1 + R + R^2 + R^3 \\
 &1 + R + R^2 + R^3 + R^4 \\
 &\quad \&c.
 \end{aligned}$$

Their sum = $\frac{n-1}{n} + \frac{n-2}{n}R + \frac{n-3}{n}R^2 \dots + \frac{R^{n-2}}{n}$ = the

amount of L. 1 annuity upon a single life, whose complement is n , at a given rate of interest.

Hence, the sum of $n-1$ terms, in Table IV. of annuities certain, divided by n , will give the amount of L. 1 annuity upon a single life whose complement is n .

Ex. Sought, the amount of L. 1 annuity during a life aged 80 years, at 4 per cent. Here $n=6$, and the five first terms of Table IV. = 15.8244
 divided by 6 = 2.64 =
 the amount sought.

Or thus: Let s denote the amount of L. 1 annuity upon a life of a given age, at a given rate, $\frac{a}{b}$ the probability that a life will continue during one year preceding to that corresponding to s ; then $\frac{a}{b} \times sR + 1$ = the amount of L. 1 annuity upon a single life, one year younger than that corresponding to s ; hence the foregoing table.

As the difference of the terms in Table IV. of annuities certain, doth constantly increase; and although the $n-1$ number of terms is divided by n , in finding the amount of L. 1 annuity upon a life whose complement is n , yet in all ages below

low 68, that amount, at 4 per cent. will be greater than that of their expectation, considered as so many years certain.

By this table, the amount of p , a sum of money, whose interest, at 4 per cent. remains unpaid during a single life of a given age, $= 1 + rs \times p$.

TABLE V. Shewing the true Probabilities of Life in London, for all ages, formed by Dr Price, from the Bills of Mortality for 10 Years preceding Anno 1769.

Ages.	Living.	Dead.	Ages.	Living.	Dead.	Ages.	Living.	Dd.	Ages.	Liv.	Dd.
0	1518	486	24	463	8	48	242	9	72	64	6
1	1032	200	25	455	8	49	233	9	73	58	5
2	832	83	26	447	8	50	224	9	74	53	5
3	747	59	27	439	8	51	215	9	75	48	5
4	688	42	28	431	9	52	206	8	76	43	5
5	646	23	29	422	9	53	198	8	77	38	5
6	623	20	30	413	9	54	190	7	78	33	4
7	603	14	31	404	9	55	183	7	79	29	4
8	589	12	32	395	9	56	176	7	80	25	3
9	577	10	33	386	9	57	169	7	81	22	3
10	567	9	34	377	9	58	162	7	82	19	3
11	558	9	35	368	9	59	155	8	83	16	3
12	549	8	36	359	9	60	147	8	84	13	2
13	541	7	37	350	9	61	139	7	85	11	2
14	534	6	38	341	9	62	132	7	86	9	2
15	528	6	39	332	10	63	125	7	87	7	2
16	522	7	40	322	10	64	118	7	88	5	1
17	515	7	41	312	10	65	111	7	89	4	1
18	508	7	42	302	10	66	104	7	90	3	1
19	501	7	43	292	10	67	97	7	91	2	1
20	494	7	44	282	10	68	90	7	92	1	1
21	487	8	45	272	10	69	83	7	93	0	0
22	479	8	46	262	10	70	76	6			
23	471	8	47	252	10	71	70	6			

By

By Table V. the number of births or burials, which are supposed to be equal, is to that of inhabitants as 1518 : 27935—759; that is, as 1 : 17.90; one out of forty who are born in London arrives at 80 years of age, and the expectation of life is as follows:

Ages.	Expectat.	Ages	Expectat.	Ages.	Expectat.	Ages.	Expect.
0	17.90	25	26.67	50	15.90	75	6.33
5	35.23	30	24.12	55	13.93	80	4.98
10	34.92	35	21.77	60	11.71	85	3.32
15	32.32	40	19.51	65	9.71	90	1.50
20	29.37	45	17.64	70	8.77		

TABLE

TABLE VI. Shewing the Value of L. 1 Annuity upon a single Life, according to the true Probabilities of Life in London.

Ages	3p.cent.	4p.cent.	5p.cent.	Ages	3p.cent.	4p.cent.	5p.c.
8-9	19.13	16.39	14.27	51	11.00	10.07	9.26
7-10	19.05	16.35	14.25	52	10.83	9.93	9.15
11	18.94	16.28	14.21	53	10.60	9.75	8.97
12	18.83	16.21	14.16	54	10.40	9.56	8.82
6-13	18.69	16.10	14.09	55	10.13	9.32	8.61
14	18.50	15.97	13.99	56	9.85	9.08	8.40
5-15	18.27	15.80	13.85	57	9.56	8.83	8.19
16	18.04	15.62	13.71	58	9.28	8.58	7.97
17	17.83	15.46	13.59	59	8.94	8.33	7.75
4-18	17.61	15.30	13.47	60	8.76	8.13	7.58
19	17.39	15.14	13.34	61	8.54	7.94	7.41
20	17.17	14.96	13.20	62	8.26	7.69	7.20
21	16.94	14.79	13.06	63	7.98	7.45	6.98
22	16.74	14.63	12.94	64	7.71	7.21	6.76
23	16.53	14.48	12.82	65	7.44	6.97	6.55
3-24	16.32	14.32	12.69	66	7.18	6.74	6.34
25	16.11	14.15	12.56	67	6.93	6.52	6.14
26	15.89	13.98	12.43	68	6.69	6.31	5.95
27	15.66	13.80	12.29	69	6.48	6.11	5.77
28	15.43	13.62	12.14	70	6.29	5.94	5.62
29	15.24	13.47	12.02	71	6.03	5.71	5.40
2-30	15.03	13.31	11.89	72	5.79	5.50	5.20
31	14.83	13.16	11.77	73	5.58	5.31	5.04
32	14.62	12.99	11.64	74	5.29	5.04	4.79
33	14.41	12.83	11.50	75	5.02	4.79	4.56
34	14.20	12.66	11.37	76	4.77	4.56	4.34
35	13.98	12.49	11.23	77	4.56	4.37	4.16
36	13.76	12.31	11.08	78	4.41	4.23	4.03
37	13.54	12.13	10.94	79	4.17	4.00	3.82
38	13.32	11.95	10.79	80	3.98	3.83	3.65
39	13.12	11.77	10.63	81	3.66	3.54	3.42
40	12.93	11.62	10.51	82	3.37	3.26	3.15
41	12.75	11.47	10.39	83	3.12	3.02	2.93
42	12.56	11.32	10.27	84	2.95	2.87	2.79
43	12.38	11.17	10.15	85	2.60	2.53	2.46
44	12.21	11.03	10.04	86	2.37	2.21	2.16
45	12.04	10.90	9.93	87	2.00	1.96	1.92
46	11.87	10.77	9.82	88	1.89	1.85	1.82
47	11.72	10.64	9.72	89	1.43	1.41	1.38
48	11.57	10.53	9.61	90	0.96	0.95	0.94
49	11.37	10.37	9.50	91	0.48	0.48	0.48
50	11.19	10.22	9.38				

Tables VI. and IX. are constructed by the theorem ${}^a_{bR} \times 1 + N$, which expresses the value of a single life, one year younger than that corresponding to N .

TABLE

TABLE VII. Shewing the Value of L. 1 Annuity upon two joint Lives, according to the true Probabilities of Life in London.

Age.	Age.	3p.cent.	4p.cent.	5p.cent.	Age.	Age.	3p.cent.	4p.cent.	5p.cent.
10	10	14.39	12.79	11.29	30	30	10.98	9.97	9.04
	15	14.09	12.54	11.10		35	10.54	9.58	8.75
	20	13.54	12.03	10.79		40	10.04	9.13	8.40
	25	12.97	11.56	10.44		45	9.57	8.74	8.10
	30	12.32	11.01	10.02		50	9.11	8.36	7.80
	35	11.68	10.50	9.60		55	8.44	7.76	7.32
	40	10.96	9.91	9.10		60	7.52	6.96	6.59
	45	10.39	9.41	8.70		65	6.58	6.10	5.81
	50	9.85	8.93	8.31		70	5.65	5.25	5.04
	55	9.01	8.26	7.74	35	35	10.14	9.30	8.38
	60	7.91	7.32	6.90		40	9.70	8.86	8.15
	65	6.86	6.37	6.04		45	9.28	8.50	7.88
15	70	5.85	5.46	5.22		50	8.88	8.15	7.61
	15	13.71	12.22	10.89		55	8.26	7.63	7.18
	20	13.06	11.79	10.57		60	7.38	6.84	6.48
	25	12.55	11.34	10.24		65	6.47	6.00	5.72
	30	11.98	10.83	9.85		70	5.58	5.18	4.97
	35	11.38	10.32	9.44	40	40	9.19	8.54	7.84
	40	10.73	9.76	8.98		45	8.86	8.23	7.61
	45	10.16	9.29	8.60		50	8.53	7.92	7.38
	50	9.60	8.83	8.22		55	7.96	7.42	6.98
	55	8.84	8.18	7.67		60	7.17	6.69	6.34
	60	7.81	7.26	6.84		65	6.33	5.89	5.62
	65	6.83	6.32	6.00		70	5.50	5.09	4.90
	70	5.81	5.43	5.19	45	45	8.61	7.94	7.33
20	20	12.76	11.47	10.26		50	8.30	7.67	7.15
	25	12.27	11.04	9.95		55	7.85	7.23	6.81
	30	11.76	10.58	9.61		60	7.05	6.53	6.20
	35	11.19	10.10	9.24		65	6.24	5.78	5.52
	40	10.58	9.57	8.81		70	5.41	5.00	4.81
	45	10.05	9.14	8.47	50	50	7.96	7.39	6.89
	50	9.50	8.69	8.10		55	7.53	7.00	6.59
	55	8.75	8.06	7.56		60	6.85	6.36	6.04
	60	7.75	7.17	6.76		65	6.09	5.65	5.40
	65	6.73	6.25	5.93		70	5.29	4.90	4.72
	70	5.76	5.38	5.15	55	55	7.16	6.67	6.23
25	25	11.87	10.74	9.66		60	6.57	6.11	5.80
	30	11.40	10.30	9.36		65	5.88	5.46	5.22
	35	10.89	9.86	9.03		70	5.12	4.75	4.59
	40	10.33	9.36	8.63	60	60	6.14	5.72	5.44
	45	9.82	8.94	8.29		65	5.56	5.18	4.97
	50	9.32	8.53	7.96		70	4.86	4.52	4.39
	55	8.62	7.94	7.46	65	65	5.22	4.81	4.64
	60	7.64	7.07	6.68		70	4.52	4.23	4.11
	65	6.66	6.18	5.88		70	4.33	4.10	3.94
	70	5.71	5.32	5.10					

TABLE VIII. Shewing the Probabilities of Life, according to the Bills of Mortality in Kettle, for 20 Years, preceding Anno 1790.

Ages	Living.	Dead	Ages	Living.	Dd.	Ages	Living.	Dd.	Ages	Living.	Dd.
0	625	20	24	392	6	48	295	5	72	104	8
1	605	40	25	386	6	49	290	6	73	96	8
2	565	36	26	380	6	50	284	7	74	88	8
3	529	20	27	374	6	51	277	7	75	80	8
4	509	16	28	368	5	52	270	7	76	72	8
5	493	12	29	363	5	53	263	7	77	64	8
6	481	6	30	358	4	54	256	7	78	56	7
7	475	4	31	354	4	55	249	7	79	49	6
8	471	4	32	350	4	56	242	7	80	43	5
9	467	4	33	346	4	57	235	8	81	38	4
10	463	4	34	342	4	58	227	8	82	34	3
11	459	4	35	338	4	59	219	8	83	31	3
12	455	4	36	334	4	60	211	9	84	28	3
13	451	5	37	330	4	61	202	9	85	25	3
14	446	5	38	326	3	62	193	9	86	22	3
15	441	5	39	323	3	63	184	9	87	19	3
16	436	5	40	320	3	64	175	9	88	16	3
17	431	5	41	317	3	65	166	9	89	13	3
18	426	5	42	314	3	66	157	9	90	10	3
19	421	5	43	311	3	67	148	9	91	7	2
20	416	6	44	308	3	68	139	9	92	5	2
21	410	6	45	305	3	69	130	9	93	3	1
22	404	6	46	302	3	70	121	8	94	2	1
23	398	6	47	299	4	71	112	8	95	1	1

By Table VIII. the number of births or burials is, to that of inhabitants, as 625 : 25368 — 312.5; that is, as 1 : 40.1; 2 out of 29 arrive at the age of 80 years, and the expectation of life is as follows:

Ages.	Expectat.	Ages	Expectat.	Ages	Expectat.	Ages.	Expect.
0	40.10	25	34.99	50	18.39	75	7.23
5	45.21	30	32.54	55	15.63	80	6.41
10	43.01	35	29.31	60	12.98	85	4.42
15	40.03	40	25.83	65	10.82	90	2.30
20	37.29	45	21.98	70	8.91	95	0.50

Ex.

Ex. Let there be two lives, whose respective ages are 40 and 30 years ; then

$$\left. \begin{array}{l} N = 25.83 \\ M = 32.54 \end{array} \right\} \text{the double of which } \left\{ \begin{array}{l} 51.66 = n. \\ 65.08 = m. \end{array} \right.$$

$$N - \frac{n}{3m} N = 19.00 = \left\{ \begin{array}{l} \overline{NM}, \text{ expectation of their} \\ \text{joint continuance.} \end{array} \right.$$

$$\left. \begin{array}{l} N - \overline{NM} = 6.83 = \\ M - \overline{NM} = 13.54 = \end{array} \right\} \begin{array}{l} \text{the expectation of} \\ \text{survivorship on the} \\ \text{part of} \end{array} \left\{ \begin{array}{l} \text{the oldest.} \\ \text{the youngest.} \\ \text{both. And} \end{array} \right.$$

$$\left. \begin{array}{l} N + M - \overline{NM} \\ = M + \frac{n}{3m} N \end{array} \right\} = 39.37, \left\{ \begin{array}{l} \text{the expectation of the} \\ \text{longest liver.} \end{array} \right.$$

TABLE IX. Shewing the Value of L. 1 Annuity upon a single Life, according to the Probabilities of Life at Kettle, computed at 4 per Cent.

Ages.	Values.	Ages.	Values.	Ages.	Values.	Ages.	Values.
6-7	18.40	30	16.41	53	10.73	76	5.19
5-8	18.29	31	16.26	54	10.49	77	5.07
9	18.19	32	16.11	55	10.21	78	5.03
10	18.08	33	15.94	56	9.93	79	4.98
4-11	17.97	34	15.78	57	9.63	80	4.90
12	17.85	35	15.60	58	9.37	81	4.73
13	17.73	36	15.42	59	9.10	82	4.54
14	17.65	37	15.23	60	8.82	83	4.18
15	17.56	38	15.03	61	8.59	84	3.81
3-16	17.47	39	14.82	62	8.35	85	3.44
17	17.38	40	14.55	63	8.10	86	3.06
18	17.29	41	14.28	64	7.86	87	2.68
19	17.15	42	13.99	65	7.62	88	2.32
20	17.05	43	13.69	66	7.38	89	1.96
21	17.00	44	13.38	67	7.14	90	1.66
22	16.94	45	13.05	68	6.91	91	1.46
23	16.88	46	12.71	69	6.68	92	1.13
24	16.82	47	12.35	70	6.47	93	0.95
25	16.77	48	12.03	71	6.26	94	0.48
26	16.72	49	11.72	72	6.02	95	0.00
2-27	16.66	50	11.45	73	5.78		
28	16.61	51	11.21	74	5.55		
29	16.51	52	10.96	75	5.35		

OF ANNUITIES

TABLE X. Shewing the Value of L. 1 Annuity on the joint Continuance of two Lives, according to the Probabilities of Life at Kettle, computed at 4 per Cent.

Values at 4 per Cent.

10	14.38												
15	14.16	13.86											
20	13.93	13.67	13.42										
25	13.80	13.53	13.34	13.12									
30	13.62	13.42	13.25	12.98	12.73								
35	13.26	13.06	12.85	12.60	12.39	11.97							
40	12.54	12.39	12.22	12.04	11.83	11.49	11.03						
45	11.44	11.32	11.19	11.06	10.89	10.62	10.27	9.72					
50	10.22	10.11	10.01	9.91	9.79	9.60	9.33	8.92	8.38				
55	9.24	9.17	9.09	9.02	8.91	8.77	8.56	8.25	7.84	7.39			
60	8.09	8.04	7.98	7.92	7.85	7.74	7.60	7.37	7.07	6.75	6.29		
65	7.07	7.04	7.00	6.95	6.90	6.82	6.71	6.55	6.34	6.10	5.75	5.38	
70	6.08	6.05	6.02	6.00	5.95	5.89	5.82	5.70	5.55	5.38	5.15	4.88	4.53
Ages	10	15	20	25	30	35	40	45	50	55	60	65	70

Tables VII. and X. are constructed by the Theorem $A - \frac{n}{3^m}A$, which expresses the number of years certain, corresponding to the value of the joint continuance of two lives, whose complements are n and m .

TABLE

TABLE XI. Shewing the Probabilities of Life derived from the Bills of Mortality in Edinburgh, Anno 1780.

Ages.	Living.	Dd.	Ages.	Living.	Dd.	Ages.	Living.	Dd.	Ages.	Liv.	Dd.
0	753	140	24	345	4	48	233	6	72	81	6
1	613	74	25	341	4	49	227	6	73	75	6
2	539	40	26	337	4	50	221	6	74	69	6
3	499	28	27	333	4	51	215	6	75	63	6
4	471	20	28	329	4	52	209	6	76	57	6
5	451	14	29	325	4	53	203	5	77	51	6
6	437	12	30	321	4	54	198	5	78	45	6
7	425	10	31	317	4	55	193	5	79	39	5
8	415	8	32	313	4	56	188	5	80	34	5
9	407	6	33	309	4	57	183	6	81	29	4
10	401	4	34	305	4	58	177	6	82	25	4
11	397	4	35	301	4	59	171	6	83	21	3
12	393	4	36	297	4	60	165	7	84	18	3
13	389	4	37	293	5	61	158	7	85	15	3
14	395	4	38	288	5	62	151	7	86	12	3
15	391	4	39	283	5	63	144	7	87	9	2
16	377	4	40	278	5	64	137	7	88	7	2
17	373	4	41	273	5	65	130	7	89	5	2
18	369	4	42	268	5	66	123	7	90	3	1
19	365	4	43	263	6	67	116	7	91	2	1
20	361	4	44	257	6	68	109	7	92	1	1
21	357	4	45	251	6	69	102	7	93	0	0
22	353	4	46	245	6	70	95	7			
23	349	4	47	239	6	71	88	7			

By Table XI. the number of births or burials is to that of inhabitants as 753:21968—376.5, that is as 1:28.67; 1 out of 22 arrives at 80 years of age, and the expectation and value of life (at 4 per cent.) are as follow.

Ages.	Expectat.	Values.	Ages.	Expectat.	Values.	Ages.	Expect.	Value.
0	28.67	11.58	35	26.52	14.46	70	8.38	6.19
5	41.83	17.26	40	23.50	13.43	75	6.42	4.85
10	41.79	17.98	45	20.74	12.43	80	4.82	3.66
15	38.85	17.14	50	18.22	11.45	85	3.10	2.34
20	35.87	16.46	55	15.54	10.23	90	1.50	0.95
25	32.82	16.00	60	12.71	8.74	92	0.50	0.00
30	29.71	15.12	65	10.45	7.47			

It is required to find the amount of the population in the city of Edinburgh and its environs, anno 1792, exclusive of Leith. The annual number of burials = 1800; which, making allowance for adventurers, and an increasing population, may be, to the births and settlers, (at 15 years of age), as 2 : 3; and these last to one another, as 3 : 1. Hence,

$$\begin{array}{rcl}
 \text{The annual number of births} & = & 2000 \\
 \text{of burials} & = & 1800 \\
 \text{of settlers} & = & 700 \\
 \hline
 \text{their sum} & = & 4500
 \end{array}$$

Therefore half this sum - = 2250
 multiplied by the mean expectation of life,

$$\text{viz. } \frac{28.67 \times 3 + 38.85}{4} - - = 31.22$$

will produce, with the addition of 755 in
 the castle, the number of inhabitants = 71,000

Table XI. is a striking evidence of the improvements in favour of life in the city of Edinburgh, during the period of half a century, from anno 1740, at which time Dr Price constructed a table of observations adapted to that city, whereby he found, agreeable to the London tables, that only 1 in 42 of all who died reached 80 years, and that the expectation of an infant's life did not exceed 18; which difference in favour of human life may arise from the encouragement given to trade in all its branches since the year 1745, whereby the bulk of the inhabitants are better fed, clothed, and lodged, have more elbow-room, and fresher air, than at any former period.

The

The Breslaw table, (Table I.), constructed by Dr Halley, as also the Norwich and Northampton tables, published by Dr Price, agree, nearly exact in point of expectation and value, with the hypothesis, and are supposed to be adapted to Europe in general; the London table (Table V.) corresponds to a large and populous capital; and Table VIII. and XI. contain probabilities of life, which will be found to be adapted, either to Scotland in general, or to Edinburgh in particular.

As annuities on lives are commonly bought or sold at so many years purchase, the values assigned in Tables II. III. &c. may also be so reckoned; at the same time, they who sell annuities have generally $1\frac{1}{2}$ or 2 years purchase of more value than specified in the tables, from purchasers whose age is 20 years and upwards.

Ex. A person residing in London, aged 30 years, wishes to purchase an annuity for life of L. 100 per annum; what must he pay for it, reckoning interest at 4 per cent.?

Here $13.31 \times 100 = \text{L. } 1331$, the purchase-money.

An annuity begins to run from the date of the event which intitles it: and in the case of a simple annuity paid in money, the annuity ceases at the term prior to the annuitant's death; but in the case of an annuity secured by land, the heirs of the annuitant are intitled to payment to the last moment of his life. In this case, by increasing the numbers in Table II. by $.2\frac{1}{2}$ of a years purchase, to 54 years of age, by $.32$, from 54 to 70, and by $.44$ from 70 and upwards, we shall obtain the present values nearly of an annuity (secured by land) of L. 1 per annum on a
single

single life, supposing the decrements of life to be equal.

As the real law, according to which human life wastes, approaches much nearer to an arithmetical than to a geometrical progression, all those theorems of Mr de Moivre are rejected which are built upon the principle, "That the decrements of life are in a constant *Ratio*," seeing they give the value of life too small.

P R O B L E M IV.

Supposing the decrements of life to be equal, it is required, to find the value of L. 1 annuity upon the longest of two lives, whose respective complements are n and m , (m being the greater number), at a given rate of interest.

64. $\frac{n-1}{n}$ is the probability that a life, whose complement is n , will exist one year, and $\frac{1}{n}$ the chance of its failing in that time; but the sum of these two fractions is equal to unity: Therefore having one of them given, we may, by subtraction, find the other. Now the probabilities of the continuance of those lives for one year, are respectively $\frac{n-1}{n}$ and $\frac{m-1}{m}$; and the probabilities of their severally failing are $1 - \frac{n-1}{n}$ and $1 - \frac{m-1}{m}$; consequently the probability of their both failing in that time, will be $1 - \frac{n-1}{n} \times 1 - \frac{m-1}{m}$, which, subtracted from unity, leaves the probability of there

there being one of them, at least, existing at the year's end. But,

$1 - \frac{n-1}{n} \times 1 - \frac{m-1}{m} = 1 - \frac{n-1}{n} - \frac{m-1}{m} + \frac{n-1 \times m-1}{nm}$, which, taken from unity, leaves $\frac{n-1}{n} + \frac{m-1}{m} - \frac{n-1 \times m-1}{nm}$, the first payment of the annuity in the parts of L. 1.

In like manner, $\frac{n-2}{n} + \frac{m-2}{m} - \frac{n-2 \times m-2}{nm}$ is the second payment of the annuity, &c.; hence, the required annuity may be expressed in the following manner :

$$\begin{aligned} & \frac{n-1}{nR} + \frac{m-1}{mR} - \frac{n-1 \times m-1}{nmR}, \\ & \frac{n-2}{nR^2} + \frac{m-2}{mR^2} - \frac{n-2 \times m-2}{nmR^2}, \\ & \frac{n-3}{nR^3} + \frac{m-3}{mR^3} - \frac{n-3 \times m-3}{nmR^3}, \text{ \&c.} \end{aligned}$$

But the series in the first and second columns, are the values of annuities on the single lives, whose complements are n and m ; and the third column is the value of an annuity upon the joint lives of the two persons; therefore,

If, from the sum of the values of the single lives of two persons, the value of their joint lives be taken, the remainder will be the value of an annuity on the longest of them.

Or thus : Let x and y represent the respective probabilities of the two lives continuing for one year; then,

$1 - x \times 1 - y = 1 - x - y + xy$, the probability of the two lives failing in that time, subtracted from
T
unity

unity, leaves $x+y-xy$, the probability of one, at least, of the two lives surviving the year; and by substituting the values of the lives in place of the probabilities thereof, (N and M being the values of the single lives, and $\overline{NM}$, that of the two joint ones,) then $N+M-\overline{NM}$ will be the value of an annuity upon the longest of them.

Ex. It is required, to find the value of an annuity of L. 1 on the longest life of two persons, whose respective ages are 35 and 25 years, allowing interest at 4 per cent. Here,

$$N+M=29.297$$

$$-\overline{NM}=11.095$$

their difference = L. 18.202 is the value required.

If the two lives are equal, (N "being the value of the two joint ones,) then $2N-\overline{N}$ will be the value of an annuity on the longest of them.

P R O B L E M V.

Supposing the decrements of life to be equal, it is required to find the value of L. 1 annuity upon the longest of three lives, (the values of the single ones being given), at a given rate of compound interest.

65. By following a process similar to that observed in the last problem, it might be shewn, that the value of an annuity upon the longest of three lives is equal to

+ the values of the three single ones,
 — the three values of the joint lives, taken two and two,

+ the value of the three joint lives. Or thus :

Let

Let x , y , and z represent the respective probabilities of the continuance of the three lives for one year; then,

$1-x \times 1-y \times 1-z = 1-x-y-z+xy+xz+yz-xyz$, will represent the probability of the three lives failing in one year, which subtracted from unity, leaves,

$x+y+z-xy-xz-yz+xyz$, the probability of one, at least, of the three lives outliving the year; and by substituting the values of the lives in place of the probabilities thereof, (NMF being the value of the three joint lives,) then $N+M+F-\overline{NM}-\overline{NF}-\overline{MF}+\overline{NMF}$ = value of the annuity.

Ex. Let the three single lives be worth 13, 14, and 15 years purchase; sought, the value of an annuity upon the longest of them. Here,

$$\begin{array}{r} 13+14+15+8.7=50.7 \\ -10.1-10.5-10.9=\underline{31.5} \end{array}$$

the value required = 19.2

PROBLEM VI.

To determine the value of an annuity of L. 1, granted for a given term of years, denoted by n , on the contingency of its ceasing upon the extinction of an assigned life, whose complement is m , at a given rate of compound interest.

66. Since every payment of the annuity depends upon the continuance of the assigned life, whose complement is m , it will be worth n terms of the series, viz.

$\frac{m-1}{mR} + \frac{m-2}{mR^2} + \frac{m-3}{mR^3} (n)$ which may be divided into two other series, viz.

T 2

$$\frac{m}{n} \times \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} (n) = P. \quad (47)$$

$$-\frac{1}{m} \times \frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3} (n) = \frac{1}{m} \times \frac{RP - np}{r} \text{ Prob. 1st.}$$

Hence $P - \frac{RP}{mr} - \frac{np}{mr} = P - \frac{RP - np}{mr}$ is the value required.

Ex. *A*, aged 40 years, engages to pay to *B* or his heirs, an annuity of L. 1 for 15 years, if he (*A*) shall live so long. Sought, the value of *B*'s annuity, reckoning interest at four per cent. Here, $n = 15$, $m = 46$, $P = 11.118$, and $p = .55526$. Hence, $11.118 - \frac{3.234}{1.84} = 9.36$, is the value of *B*'s annuity.

TABLE XII. Shewing the Value of L. 1 Annuity for a certain Number of Years, depending upon the contingency of a given Life.

Ages, computing interest at 4 per cent.

Yrs.	10	15	20	25	30	35	40	45	50	55	60	65	70
5	4.28	4.27	4.25	4.24	4.22	4.20	4.17	4.14	4.09	4.03	3.95	3.83	3.64
10	7.56	7.52	7.47	7.42	7.36	7.29	7.20	7.09	6.94	6.76	6.50	6.11	5.49
15	10.05	9.97	9.89	9.79	9.67	9.53	9.36	9.14	8.87	8.50	8.00	7.27	
20	11.95	11.83	11.70	11.54	11.36	11.14	10.87	10.54	10.12	9.56	8.78		
25	13.36	13.20	13.02	12.81	12.56	12.26	11.89	11.43	10.85	10.08			
30	14.42	14.22	13.98	13.71	13.39	13.01	12.55	11.97	11.23				
35	15.20	14.95	14.66	14.34	13.96	13.50	12.93	12.24					
40	15.76	15.48	15.15	14.77	14.32	13.79	13.13						
45	16.27	15.85	15.56	15.05	14.54	13.92							
50	16.45	16.00	15.68	15.21	14.65								

Mr Simpson, in his select exercises, p. 262, is very defective in the construction of this table; which yet may easily be calculated from the relation which the above theorem hath to $P - \frac{RP - np}{nr} = N$, the value of the single life, whose complement is n , Prob. I. Thus, for

Ex.

Ex. A, aged 25, enjoys an annuity for the space of 20 years, if he shall live so long; it is required, to find the present value of said annuity, reckoning interest at 4 per cent. Here $n=20$, and $m=61$, the complement of the life; therefore,

From the value of an annuity for 20 years = 13.590
 Subtract the value of a life aged 66 years = 7.333

their difference = 6.257
 multiplied by $n = 20$

their product divided by $m = 61$) 125.140 $\left(\begin{array}{l} 13.59 \\ 2.05 \end{array} \right.$

There remains the answer sought = 11.54

A Question of Insurance on Lives.

A, aged 16 years, is assured of the sum of L. 500, in case he shall live to be 21 years of age complete; it is required, to find what premium he ought to pay to insure his life during these 5 = n years, upon that sum.

Let $\frac{a}{b} = \frac{65}{70}$ } = A's chance of { living 5 years.
 $\frac{n}{b} = \frac{5}{70}$ } { dying in that time.

But A's interest in L. 500 = $\frac{a}{b} \times 500 = \text{L. } 464.3$,
 their difference, viz. L. 35.7, is the premium of insurance; or A may lay his account to lose $\frac{n}{b} \times 500 = \text{L. } 35.7$, the same premium.

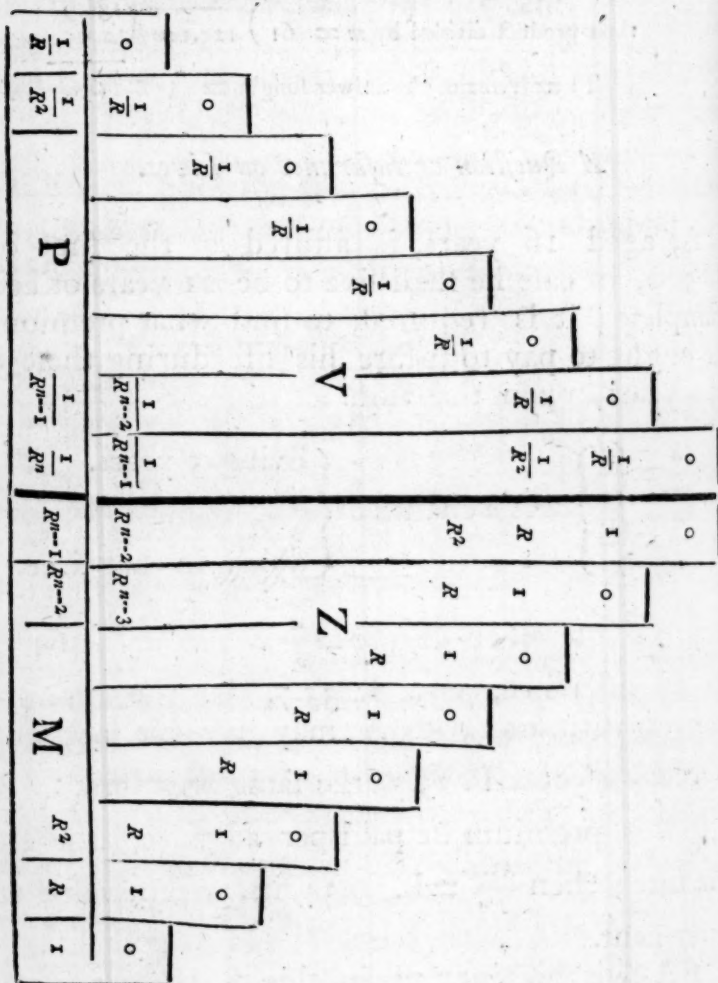
If this premium be paid per advance, at a given rate; then $\frac{35.7}{R^n} = \text{L. } 29.34$, its present value at 4 per cent.

If A sells his interest in this L. 500, at a given

ven rate; then $\frac{464.3}{R^n} = \text{L. } 382.45$, the value at 4 per cent. See Prob. XV. Ex. 1st and 2d.

Supplement to Problem I. II. III. IV. V. VI. Chap. VIII.

ART. I. To find, by a geometrical figure, the value or amount of L. 1, or of L. 1 annuity, upon a single life, whose complement is n , at a given rate of interest.



1st, Seeing, that out of n persons alive, at a given age, whose complement is n , one is supposed to fail yearly, till they are all exhausted at the age of 86; the two triangles V and Z, which contain the value or amount of L. 1 annuity for $1+2+3\ldots+n-1$ years, will represent the value or amount of L. 1 annuity upon n lives, whose common complement is n , at a given rate; and the two bottom columns P and M, which contain the value or amount of L. 1 for $1+2+3\ldots+n$ years, will shew the value or amount of L. 1 annuity for n years, at the same rate; hence,

$$\left. \begin{array}{l} \text{V or Z} = \text{val. or am. of L. 1 annuity for } 1+2+3\ldots+n-1 \text{ years,} \\ \text{P or M} = \text{— or — of ————— for } n \text{ years; then,} \\ \text{V or Z} \\ \text{or M} \end{array} \right\} \div n = \left\{ \begin{array}{l} \text{val. or am. of L. 1 annuity} \\ \text{— or — of L. 1} \end{array} \right\} \left\{ \begin{array}{l} \text{upon a single life.} \\ \text{whose complet. is } n. \end{array} \right.$$

2^{dly}, But seeing that $\frac{1-rN}{R}$ is also = value, and $1-rs$ = amount of L. 1 upon a single life, whose complement is n , at a given rate:

$$\text{Hence } N = \frac{n-RP}{rn} = \text{value} \left\{ \begin{array}{l} \text{of L. 1 annuity up-} \\ \text{on a life, whose} \end{array} \right. \\ S = \frac{M-n}{rn} = \text{amount} \left\{ \begin{array}{l} \text{complement is } n. \end{array} \right.$$

Ex. It is required, to find the value or amount of L. 1, or of L. 1 annuity, upon a life aged 36 years, at 4 per cent. Here $n=50$, $P=21.482$, $M=152.667$, $R=1.04$, and $r=.04$. Hence,

$$\left. \begin{array}{l} \frac{RP}{n} \\ \frac{P}{M} \end{array} \right\} \div rn = \left\{ \begin{array}{l} 13.829 = N, \text{ the va. } \\ 51.34 = s, \text{ the am. } \end{array} \right\} \left\{ \begin{array}{l} \text{of L. 1 an-} \\ \text{nunity.} \end{array} \right.$$

$$\left. \begin{array}{l} P \\ M \end{array} \right\} \div n = \left\{ \begin{array}{l} .4296 = \text{va.} \\ 3.053 = \text{am.} \end{array} \right\} \left\{ \begin{array}{l} \text{of L. 1, upon a life,} \\ \text{age 36 years.} \end{array} \right.$$

ART.

ART. 2. To find the value or amount of L. 1 or L. 1 annuity, upon the joint continuance of 2, 3, 4... m lives, at a given rate of interest. Here,

1st, Let n represent the mean complement of all the lives,

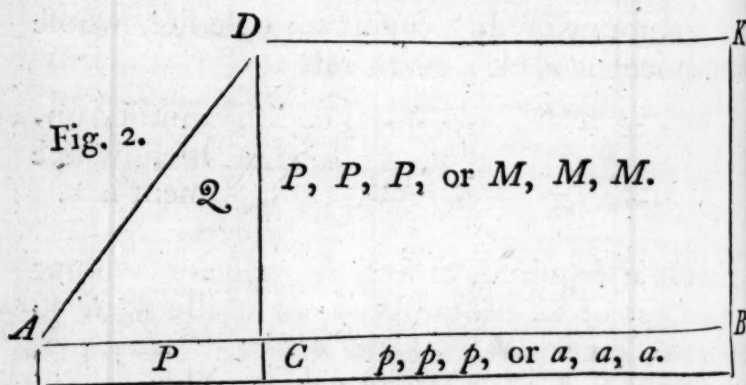
x — the expectation of their joint continuance,

X — the expectation of the longest liver,

P or M — the value or amount of L. 1 annuity for x years,

$\mathcal{Q}$ or A — the value or amount of L. 1 annuity upon a life whose complement is x , and

p or a — the value or amount of L. 1 for x years; then,



In Fig. 2. put $AB = n$, AC or $CD = x$, of consequence CB or $DK = n - x = X$, and complete the rectangle BD , and triangle $\mathcal{Q}$. Now, seeing that out of n lives, whose mean complement is n , one will fail yearly during x years, $DB + \mathcal{Q} = \frac{n - x}{n} \times P + \text{value of a life whose complement is } x \text{ multiplied}$

multiplied by x , will represent the value of L. 1 annuity upon n lives for x years; and the bottom column, viz. $n - x \times p + P$, will shew the value of L. 1 on n lives for x years; hence,

$$\frac{n-x}{=X} \left\{ \begin{array}{l} \times P \text{ or } M + x \times \mathcal{Q} \text{ or } A \\ \times p \text{ or } a + P \text{ or } M \end{array} \right\} \div n = \left\{ \begin{array}{l} \text{Val. or am. of} \\ \text{L. 1 annuity.} \\ \text{Val. or am. of} \\ \text{L. 1 during the joint continuance of the lives.} \end{array} \right.$$

Ex. It is required, to find the value or amount of L. 1, or of L. 1 annuity, upon the joint continuance of three lives, whose respective ages are 40, 36, and 26 years, reckoning interest at 4 per cent. Here,

$n = 52$, $x = 12.88$, $X = 39.12$, $P = 9.913$, $M = 16.434$, $\mathcal{Q} = 5$, $A = 6.9$, $p = .6035$, and $a = 1.657$. Hence,

$$\left. \begin{array}{l} 387.796 + 64.4 \\ 642.898 + 88.87 \\ 23.609 + 9.913 \\ 64.822 + 16.434 \end{array} \right\} \div 52 = \left\{ \begin{array}{l} 8.696 = \text{val.} \} \text{of L. 1} \\ 14.072 = \text{am.} \} \text{ann.} \\ .6446 = \text{val.} \} \text{of L. 1} \\ 1.563 = \text{am.} \} \text{du-} \\ \text{ring the joint continuance of these three lives.} \end{array} \right.$$

2d, In the case of m equal lives, whose common complement is n , $x = \frac{n}{m+1}$, and the co-efficients of P and $\mathcal{Q}$, viz. $\frac{n-x}{n} + \frac{x}{n} = \frac{mn}{mn+n} + \frac{n}{mn+n}$. Hence,

$$\frac{m \times P \text{ or } M + \mathcal{Q} \text{ or } A}{- \times p \text{ or } a + \frac{P \text{ or } M}{n}} \div m + 1 = \left\{ \begin{array}{l} \text{val. or am. of L. 1 ann.} \\ \text{value or amount} \end{array} \right.$$

of L. 1 upon the joint continuance of these lives.

In the case of 4, 5, . . . m unequal lives, find their mean age, and consider it as the age of so many equal lives.

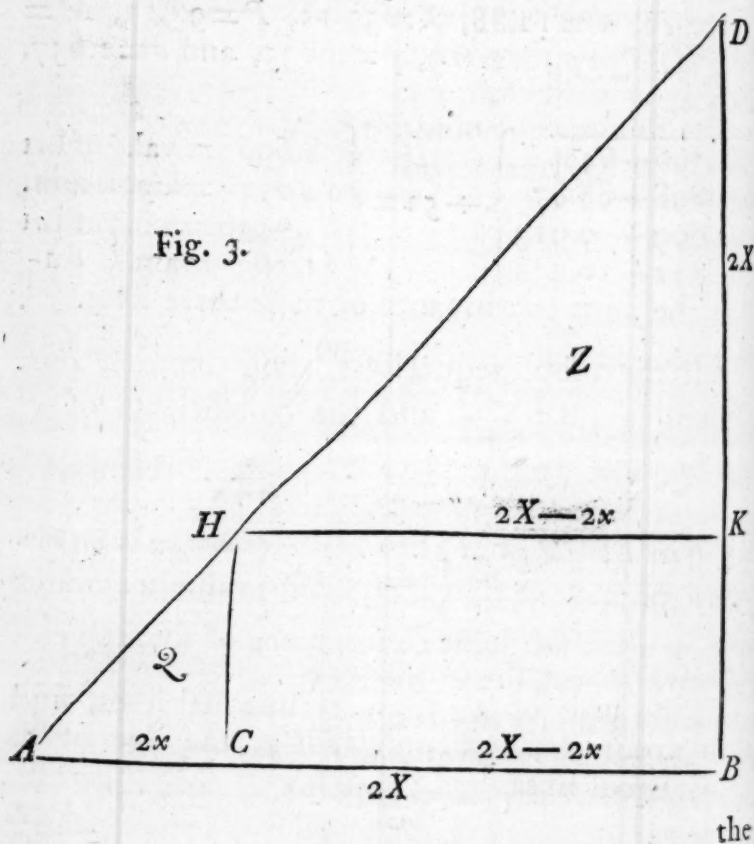
ART. 3d, To find the present value or amount of L. I, or of L. I annuity, during the longest liver of two or more lives, at a given rate of interest.

1st, Let n represent the mean complement of all the lives,

x — the expectation of their joint continuance, and

— X — the expectation of the longest liver; then

In Fig. 3. make AB or $BD = 2X$, AC or $CH = 2x$, of consequence, CB or $HK = 2X - 2x =$ twice



the expectation of survivorship, and complete the triangle *DAB*. Here,

$$\begin{array}{l} \text{The triangle } DAB \left\{ \begin{array}{l} \text{val. or am.} \\ \text{of a life} \\ \text{whose com-} \\ \text{plement is} \end{array} \right. \left\{ \begin{array}{l} 2X = \text{val. or am.} \\ \text{of longest liver.} \\ 2x = \text{val. or am. of} \\ \text{joint continuance.} \\ 2X - 2x = \text{value or} \\ \text{amount of survi-} \end{array} \right. \end{array}$$

vorship, computed at the dissolution of the joint lives, which value, $\times \frac{1-r^2}{R}$, and added to that of 2 , will also give the value of the longest liver, nearly exact. And the amount of 2 , $\times 1+rs$, and added to that of Z , will likewise shew the amount of the longest liver, nearly.

Ex. It is required, to find the value and amount of L. 1 annuity upon the longest liver of three lives, whose respective ages are 40, 36, and 26 years, at 4 per cent. Here,

$$n=52, x=12.88, X=39.12; \text{ hence,}$$

$$\begin{array}{l} \text{The triangle } DAB \left\{ \begin{array}{l} = 17.076 = \text{val.} \\ \text{by Art. 1.} \end{array} \right. \left\{ \begin{array}{l} \text{of L. 1 an-} \\ \text{nuity upon} \end{array} \right. \\ \left\{ \begin{array}{l} = 138.9 \\ \text{the longest liver of these lives.} \end{array} \right. \end{array}$$

$$\begin{array}{rcl} 2d, \text{ To the value of } 2 & = & 8.962 \\ \text{add that of } Z, \times \frac{1-r^2}{R} & = & 8.813 \\ \hline \text{Their sum} = \text{value of the longest liver} & = & 17.775 \end{array}$$

$$\begin{array}{rcl} \text{And to the amount of } Z=s & = & 56.42 \\ \text{add that of } 2, \times 1+rs & = & 56.78 \\ \hline \text{Their sum} = \text{amount of the longest liver} & = & 113.20 \end{array}$$

In Fig. 4. put $AB=2n$, EB or $BD=2X$, FB or $BC=n$, and complete the square HB , the rectangle KB , and the triangles DEB , and CFB , and join KF . Now it is evident, that the value or amount of L. 1 annuity upon the longest liver of any number of lives, where the mean complement of all the lives is n , will vibrate betwixt the triangle $\mathcal{Q}$ and square HB ; and that the line KF , and point K , will limit these values; hence,

$$\frac{\frac{2X-n}{X-x} \times P \text{ or } M + \left\{ \frac{2n-2X}{2x} \right\} \times \mathcal{Q} \text{ or } A}{\frac{2X-n}{X-x} \times p \text{ or } a + \frac{2x}{n} \times P \text{ or } M} \div n = \begin{cases} \text{val. or am. of} \\ \text{L. 1 annuity.} \\ \text{val. or am. of} \\ \text{L. 1 upon the} \\ \text{longest liver of these lives.} \end{cases}$$

Ex. In the example to this article, $n=52$, $x=12.88$, $X=39.12$, $P=21.748$, $M=167.165$, $\mathcal{Q}=14.126$, $A=55.37$, $p=.1301$ and $a=7.687$. Hence,

$$\left. \begin{array}{l} 570.667 + 363.686 \\ 385.41 + 1426.33 \\ 3.414 + 10.774 \\ 201.707 + 82.811 \end{array} \right\} \div 52 = \begin{cases} 17.968 = \text{val. of L. 1 annui-} \\ 111.783 = \text{am. ty.} \\ .2728 = \text{val. of L. 1, du-} \\ 5.471 = \text{am. ring the longest} \end{cases}$$

liver of three lives, whose respective ages are 40, 36, and 26 years, at 4 per cent.

4th, In the case of m equal lives, whose common complement is n ; $x = \frac{n}{m+1}$, $X = \frac{mn}{m+1}$,

P = value of L. 1 annuity for n years, &c, and the co-efficients of P and $\mathcal{Q}$, viz. $\frac{2X-n}{n} + \frac{2n-2X}{n}$

$$= \frac{mn-n}{mn+n} + \frac{2n}{mn+n}. \text{ Hence,}$$

$$\frac{\frac{m-1}{m+1} \times P \text{ or } M + \frac{2}{m+1} \times \mathcal{Q} \text{ or } A}{\frac{m-1}{m+1} \times p \text{ or } a + \frac{2}{n} \times P \text{ or } M} \div m+1 = \begin{cases} \text{value or amount of L. 1} \\ \text{annuity.} \\ \text{value or amount of L. 1} \\ \text{upon the longest liver of these equal lives.} \end{cases}$$

Ex.

Ex. Sought, the value of the longest liver of 4 equal lives, whose common age is 30 years, at 4 per cent. Here,

$$m=4, P=22.2198, \text{ and } Q=14.684. \text{ Hence,}$$

$$\frac{3 \times 22.2198 + 2 \times 14.684}{5} = 19.205 = \text{value required.}$$

In the case of 4, 5... m unequal lives, find their mean age, and consider it as the age of so many equal lives.

5th, If the value, &c. of an annuity be required, which shall continue during the longest liver of q out of m lives of a given age and at a given rate; here $X = \frac{qn}{m+1} = \text{expect. of the longest liver.}$

P = value of an annuity for n years, &c. and

$$\left. \begin{array}{l} \frac{q - \frac{m+1}{2}}{2} \times P \text{ or } M + \frac{m+1}{2} - q \times Q \text{ or } A \\ \text{---} \times p \text{ or } a + \frac{m+1}{2} - q \times P \text{ or } M \end{array} \right\} \div \frac{m+1}{2} = \left\{ \begin{array}{l} \text{value or am.} \\ \text{of L. 1 ann.} \\ \text{value or a-} \\ \text{mount of L. 1} \end{array} \right.$$

during the longest liver of q out of m lives.

Ex. It is required to find the value of L. 1 annuity upon the longest liver of 10, out of 15 equal lives, whose common age is 22 years, at 4 per cent. Here,

$$m=15, q=10, P=22.968 \text{ and } Q=15.669; \text{ hence,}$$

$$\frac{2 \times 22.968 + 6 \times 15.669}{8} = 17.494, \text{ the value required.}$$

When q is less than $\frac{m+1}{2}$, the first term is negative.

ART. 4. To find the value or amount of L. 1, or of L. 1 annuity, for a term of years denoted by n , but subject to failure on the contingency of

of an assigned life, whose complement is m , at a given rate of interest.

In fig. 2. suppose $AB=m$, AC or $CD=n$, and let the rectangle DB and triangle Q be completed; now, seeing that out of m lives, whose complement is m , one is supposed to fail yearly during n years, let

P or M = value or amount of L. 1 annuity for n years.

Q or A = — or — of an annuity upon a life whose complement is n .

p or a = — or — of L. 1 for n years; then

$$\frac{m-n \times P \text{ or } M + n \times Q \text{ or } A}{- \times p \text{ or } a + P \text{ or } M} \div m = \left\{ \begin{array}{l} \text{val. or am of L. 1 annuity.} \\ \text{— or — of L. 1 for a} \end{array} \right.$$

term of years denoted by n , on the contingency of an assigned life whose complement is m .

Ex. It is required, to find the value or amount of L. 1, or of L. 1 annuity, during the term of 15 years, yet liable to be demanded, or subject to failure, on the contingency of a life aged 36 years, at 4 per cent.

Here $n=15$, $m=50$, $P=11.1184$, $M=20.0236$, $Q=5.728$, $A=8.38$, $p=.5553$, and $a=1.8$. Hence,

$$\left. \begin{array}{l} 339.144 + 85.92 \\ 700.825 + 125.70 \\ 19.435 + 11.118 \\ 63.000 + 20.024 \end{array} \right\} \div 50 = \left\{ \begin{array}{l} 9.501 = \text{value} \\ 16.53 = \text{amount} \end{array} \right\} \text{ of L. 1 annuity.}$$

$$\left\{ \begin{array}{l} .611 = \text{value} \\ 1.66 = \text{amount} \end{array} \right\} \text{ of L. 1 during the term of 15 years, on the contingency of a life aged 36.}$$

REMARKS.

REMARKS.

1st, The values, &c. acquired by Art. 1. and 4. are perfectly accurate, and agree exactly with the algebraic investigations, Prob. I. and VI.

2^d, The value, &c. of the longest liver, found by Art. 3. is discovered with much less labour, and is nearly as accurate as the results of Prob. IV. and V.

3^d, The value, &c. of 3, 4 . . . m joint lives, Art. 2. is as perfect as the nature of the science will admit. The value of 2 joint lives is nearly correct at the early periods of life ; but it declines as life advances, in such a manner, that, at the mean age of 40, it would require 1 year's purchase to be added to $\mathcal{Q}$, at 4 per cent. to make it correspond with Prob. II. and III.

4th, The theorems contained in these articles, (excepting case 2^d, Art. 1.), may with equal propriety be applied to life-annuities at simple interest, by making use of tables of simple, in place of those of compound interest. They may also be used in finding the value of life from a table of observations.

Ex. 1st, To find, from the Fifth Table of Observations, the value of the joint continuance of two lives, whose respective ages are 30 and 25 years, at 4 per cent. Here,

$N =$

$$\begin{array}{lcl}
 N=32.54 & x=N^{\circ} \text{ failing in } \overline{NM} \text{ years} & =15.84 \\
 M=34.99 & n-x & =51.69 \\
 \text{Sum}=n=67.53 & P=\text{val. of ann. for } \overline{NM} \text{ yrs} & =14.63 \\
 \overline{NM}=22.45 & Q=\text{val. of a life } \text{---} & =8.05 \text{ and} \\
 \frac{51.69 \times 14.63 + 15.84 \times 8.05}{67.53} & =13.08, \text{ value sought;} & \\
 \text{which agrees nearly exact with the result of the} & & \\
 \text{theorem } \frac{a}{b} A. & &
 \end{array}$$

Ex. 2d, To find the value of the longest liver of two lives, whose respective ages are 40 and 30 years, at 4 per cent. from the same table. Here,

$$\begin{array}{lcl}
 N=25.83 & 2X-n & =20.37 \\
 M=32.54 & 2n-2X & =38.00 \\
 \text{Sum}=n=58.37 & P=\text{val. of ann. for } n \text{ years} & =22.466 \\
 \overline{NM}=19.00 & Q=\text{val. of a life aged } 35 & =15.60 \\
 \text{Diff.}=X=39.37 & \text{Hence,} &
 \end{array}$$

$$\frac{20.37 \times 22.466 + 38 \times 15.6}{58.37} = 18.00, \text{ value;} \text{ which is the}$$

same with the result of the theorem $X - \frac{x}{N} \times d$,

Prob. III. Case 2.

OF REVERSIONS.

PROBLEM VII.

67. **T**O find the value of the reversion of an estate in fee simple, after one or more lives of given ages, at a given rate of compound interest. Here,

$$\text{Or, } \frac{1}{r} - \frac{1}{r} - \frac{RP}{nr} = \frac{RP}{nr} \left| \begin{array}{l} = \text{the reversion, in per-} \\ \text{petuity, of L. 1 per an-} \\ \text{num, after a life whose} \\ \text{value is } N, \text{ at a given} \\ \text{rate of interest.} \end{array} \right.$$

Ex. Sought, the value of a reversion, in perpetuity, of L. 1 per annum, after the life of a person aged 43 years, allowing interest at 4 per cent. Here,

$$\frac{1}{r} - N = \frac{RP}{nr} = 25 - 12.683 = 12.317, \text{ the reversion.}$$

In like manner,

$$\left. \begin{array}{l} \frac{1}{r} - \overline{NM} \\ \frac{1}{r} - \overline{NMF} \end{array} \right\} = \text{a reversion in perpetuity after } \left\{ \begin{array}{l} 2 \text{ joint lives} \\ 3 \text{ joint lives} \end{array} \right.$$

If the purchase is discharged in yearly payments during the continuance of the life or lives in question; in this case, the reversion, divided by the value of the life or lives, will quot the annual payments.

Ex. A and B, whose respective ages are 35 and 25 years, sell the reversion of an estate whose rent is L. 150 per annum, for yearly payments during the longest of their lives; sought, these payments, reckoning interest at 4 per cent. Here,

$$\begin{array}{r} 25 - 18.202 = 6.798 \\ \text{multiplied by } 150 \end{array}$$

And the product divided by 18.2) 1019.7 (L. 56, the annual payment.

If the first of the annual payments be made per advance, the above divisor, viz. 18.2, increased by unity, will quote L. 52.1, the annual payments.

Cor. 1st, To find the present value of L. 1, due at the decease of a person of a given age, at a given rate of interest.

L. 1, or any other sum of money, may be conceived to be the present worth of a perpetuity, whose rent is equal to the interest of that sum for one year; hence, the reversion of a fee-simple equal to that interest, after the given life, will be the value required. Thus,

$\frac{1}{r} - N \times r = 1 - rN = \frac{RP}{nr} \times r = \frac{RP}{n}$ would be the value required, were it not for one circumstance, viz. that, from the nature of the calculation, the time in which the first yearly payment of a reversionary annuity becomes due, is the end of the year in which the event happens that entitles it. This is also the time when a reversionary sum becomes due. But an annuity, the first payment of which is to be made at the same time with

X 2

another

another payment of a sum in hand sufficient to buy an equal annuity, is worth one year's purchase more than the sum. Hence,

$\frac{1-rN}{R} = \frac{p}{n}$ = the value of L. 1 due at the decease of a person whose complement of life is n .
Or thus:

$\frac{1}{n}$ = the present probability, that a life whose complement is n will fail in any one assignable year of its duration; (see Prob. XIX. case 1.)
Hence,

$\frac{1}{nR} + \frac{1}{nR^2} + \frac{1}{nR^3} (n) = \frac{p}{n}$ = the present value of L. 1, payable at the failure of a life whose complement is n ; which, multiplied by s , a sum due at the close of a given life, will produce the present value thereof.

Ex. Let there be a legacy of L. 100 = s , due at the decease of a person aged 60 years; sought, its present value, at 4 per cent. Here,

$s \times \frac{p}{n} = 100 \times .6147 = \text{L. } 61.47$, the value required.

Cor. 2d, To find what sum ought to be paid, on the decease of a person of a given age, in consideration of L. 1 now received. It is evident, that the reciprocal of the former corollary will be the answer in this. Thus,

$\frac{R}{1-rN} = \frac{n}{p}$ = the sum which ought to be paid at the decease of a person whose complement of life is n , for L. 1 now received, which, multiplied by p ,

p , any present sum, will produce s , the amount thereof at the close of a given life. Or thus:

Seeing $\frac{1}{n} + \frac{R}{n} + \frac{R^2}{n}$, $(n) = \frac{M}{n}$, the amount of L. 1 annuity for n years certain divided by n , and expresses the amount of L. 1, payable at the failure of a life whose complement is n ; therefore,

$$p \times \frac{M}{n} = s.$$

Ex. Let $p = \text{L. } 100$, the age 60 years, and the rate of interest 4 per cent.: Here, $n = 26$, $M = 44.3117$, and $p \times \frac{M}{n} = 100 \times '1.7043 = \text{L. } 170.43 = s$, which is the same with the result from the theorem $p \times \overline{1+rs}$ of Table IV.

PROBLEM VIII.

68. Supposing the decrements of life to be equal, it is required, to find the value of the reversion of an annuity, which is to continue during the life of a person whose complement is m , after the decease of the present possessor, whose complement of life is n , at a given rate of interest.

$1 - \frac{n-1}{n}$, $1 - \frac{n-2}{n}$, &c. are the probabilities of the possessor's dying in the first, second, &c. years; and $\frac{m-1}{m}$, $\frac{m-2}{m}$, &c. are the probabilities of the expectant's living to the end of the first, second, &c. years. Hence $\frac{m-1}{m} \times 1 - \frac{n-1}{n}$, $\frac{m-2}{m} \times 1 - \frac{n-1}{n}$, &c. are the probabilities that the expectant

tant will receive the first, second, &c. yearly payment, which probabilities being expanded by multiplication, and their present values found, will give the value of the reversion, viz.

$$\frac{m-1}{mR} - \frac{\overline{m-1} \times \overline{n-1}}{mnR},$$

$$\frac{m-2}{mR^2} - \frac{\overline{m-2} \times \overline{n-2}}{mnR^2}, \text{ \&c.}$$

But the series in the first column is the value of the expectant's life, whose complement is m ; and the second column contains the value of the joint lives of the possessor and expectant. Therefore, if, from the value of the expectant's life, the value of the joint lives be taken, the remainder will be the value of the reversion.

Or thus: Let $\overline{1-x} \times y = y - xy$ be the probability that the expectant will outlive the possessor the first year; then, by substituting the values of the lives in place of the probabilities thereof, $M - \overline{NM}$ will be the present value of the expectant's life after the death of the possessor, or the value of the reversion required.

Ex. Let the respective ages of the expectant and possessor be 25 and 30 years, and the rate of interest 4 per cent. then $M = 15.318$, $N = 14.684$ and $\overline{NM} = 11.468$; hence, $M - \overline{NM} = 3.85$, the value of the reversion.

Of a Widows Scheme.

69. **T**HE resolution of the several questions respecting a reversionary annuity to one life after two, is derived from the above theorem, viz. $M - \overline{NM}$ = the value of the reversion. There are two cases of such an annuity, which differ from or agree with one another in a few particulars.

Case 1. A and B, whose respective ages are 30. and 25 years, agree to pay a certain sum in hand, or a certain annual rate during the continuance of their joint lives, in order to purchase an annuity to B after A's decease. Sought these, reckoning interest at 4 per cent.

$M - \overline{NM} = 3.85$ = sum in hand, or present value.

$\frac{M - \overline{NM}}{\overline{NM}} = L.0.3357$ = annual rate payable during their joint lives to secure L. 1 annuity to B.

In this case, the payment of the rates ceases at the close of the joint lives, whether A be alive or not, and B is supposed to be the survivor.

Had this case been conditional, viz. "if B shall survive A," the sum in hand and annual rate must be diminished by $\frac{41.57}{61.00} = .681$, the probability of B's surviving the dissolution of marriage.

Had part of the present value of the annuity, such as L. 1.5 been paid in ready money, the remainder, divided by $\overline{NM}$, quotes the annual rate; and were the rates payable per advance, $\overline{NM} + 1$ will be the divisor.

Case 2d, Is that of a widows scheme, where
either

either a certain sum in hand, or an annual rate during the life of A, is paid, in order to secure a provision for B, after A's decease, or, failing B, so many years purchase of her annuity to their children.

This case differs *in toto* from the conditional one mentioned above; and in part from Case 1st, in this, that the annual rate is paid during the life of A; but agrees with it in supposing that either a widow or children are left a burden upon the scheme. Therefore,

Let the respective ages of A and B be 30 and 25 years, and the rate of interest 4 per cent; then,
 $M - \overline{NM} = L. 3.85 =$ sum in hand, or present value.

$\frac{M - \overline{NM}}{N} = L. 0.2622 =$ annual rate, exclusive of the expence of management, payable during the life of A, to secure L. 1 annuity to B, after A's decease, or, failing B, 10 years purchase of her annuity to their children.

This, I acknowledge, is different from the usual method of constructing a widows scheme, and in particular from that recommended by Dr Price, whereby, to find the annual rate, the present value of the annuity must be divided by that of the joint lives, as in Case 1st; making the full value of the annuity, in the present instance, to be paid in 19, whereas A hath a chance of living 28 years: Besides,

$$\begin{array}{l} L. 3.85 \text{ val.} \\ L. .2622 \text{ ann.} \\ L. .3357 \text{ ann.} \end{array} \left\{ \begin{array}{l} \text{accumulated during} \\ \text{A's life, by Tab. IV.} \end{array} \right. = \left\{ \begin{array}{l} L. 13.74 \\ L. 16.83 \\ L. 21.55 \end{array} \right. \text{ and} \\ \text{val. of B's ann. at widowity, per said tab.} = L. 14.20$$

The above rate or value, multiplied by any annuity

annuity, will, with the addition of the expence of management, shew the rate or value corresponding to that annuity. the ages being 30 and 25. Thus,

<i>Exp.</i>	<i>Ann. Rate.</i>	<i>Value.</i>	<i>Exp.</i>
.25 +	10 = 2.872	L. 38.5	L. 25 p.an.
.375 +	15 = 4.308	57.75	.375
.5 + .2622 ×	20 = 5.744	77.0 with	.5
.625 +	25 = 7.180	96.25	.625
.75 +	30 = 8.616	115.5	.75

Again, let the respective ages of A and B be 40 and 30 years, the annuity L. 20, and the rate of interest 4 per cent; then $4.256 \times 20 = \text{L. } 85.12$ = value of the annuity; and, $.5 + .3225 \times 20 = \text{L. } 6.95$ = annual rate. But the number at the former ages may be to that at the latter, as 3 : 1; thence $\frac{5.744 \times 3 + 6.95}{4} = \text{L. } 6.05$, the mean annual rate corresponding to L. 20 annuity.

Had these rates been constructed upon the principles of the expectation of life, as in the Church's Scheme; in this case, the present value of an annuity for 20 years, at 4 per cent. = 13.590

divided by the amount thereof for 30 years = 56.085
 quotes the annual rate = L. .2423, which is less than the above. In such a scheme, in order to assist their operation, and to balance the advantage of the marriage-tax in the Church's Scheme, the rates might be paid per advance;—the widow's annuity may begin to run from the first term of Whitsunday or Martinmas after the decease of the contributor;—where there is no widow, 10 years purchase of her annuity might be granted to the children of a contributor, with-

in a twelvemonth after his decease;—if a contributor, before his death, hath not paid 10 annual rates, a deduction might be made out of the provision of the widow or children, till the balance be recovered;—and for each 500 contributors, the annual expence of management ought not to exceed L. 110.

Supposing then, that there were 500 married contributors in this scheme, of 30 years of age and upwards, whose number being constantly the same,

Their expectation of life = 32.5 years

The number dying yearly = 15.0

of whom, 11 will leave widows, 3 of them, families of children without a widow, and 1, neither a widow nor children.

The annual number of marriage taxes, each at L. 9,

= 3.0

The expectation of widowity,

= 19.5

The maximum of annuitants,

= 214.0

of whom 203 will draw full, and 11, including heirs, half annuities; and the families of children,

= 3.0

Deductions out of the provision of widows or children,

= L. 82.6

The medium of annuities L. 20, and of rates,

L. 6.05

The annual expence of management = L. 110

And the capital necessary to support the scheme, when the annuitants shall arrive at a maximum,

= L. 41,390

Of a Marriage-Tax.

70. A marriage-tax may be considered as the difference

ference in point of value, of the two Widows Annuities, computed at the commencement of the second marriage: Thus,

Supposing the above contributor becomes a widower, and at the age of 40 marries a second wife, whose age is 30 years; sought, his marriage-tax, at 4 per cent.

From the value of two joint lives aged 40
and 30 years, = L. 4.256

Subtract that of two joint lives aged 40
and 35 years, = 3.822

The remainder, = .434

Multiplied by 20, will produce the tax for
that annuity, = L. 8.68

Quest. A and B, whose respective ages are 60 and 40 years, are subjected to the burdens and enjoy the advantages of a widows scheme, where in B's annuity, should she survive A, is L. 25; B, designing to sell her annuity, desires to know the present value thereof, reckoning interest at 4 per cent.

The expectation of the joint continuance of
these two lives, = 10.55

And B's probability of surviving the dissolution of marriage is of consequence, $\frac{35.45}{46.00} = .77$

But $M - \overline{NM} = 5.686$ would be the years purchase of the annuity, were it certain that B would be the survivor. Hence,

$.77 \times 5.686 \times 25 = \text{L. } 109.455$, is the value required.

Or thus, A's expectation of life = 13, and B's probability of surviving A = $\frac{m - \frac{1}{2}n}{m} = \frac{33}{46} = .717$;
Y 2 hence,

hence, B's age at A's death would be 53 years, whose value, viz. 10.702, reduced for 13 years, at 4 per cent, gives 6.43, the years purchase of the annuity, were it certain that she would survive A; hence, $.717 \times 6.43 \times 25 = \text{L. } 115.25$, is the value required.

PROBLEM IX.

It is required, to find the value of the reversion of an annuity, which is to continue during the joint lives of two persons, after the decease of the present possessor, at a given rate of compound interest.

71. Let x, y , and z be the respective probabilities of the continuance of the three lives for one year; then $1-z$ will be the probability of the possessor's life failing the first year; which, multiplied by xy , gives $xy - xyz$, the probability that the two expectants will outlive the possessor the first year. Hence, by substituting the values of the lives in place of the probabilities thereof,

$$\overline{NM} - \overline{NMF} = \text{the value of the reversion.}$$

Therefore, if, from the value of the two joint lives in expectation, the value of the three joint lives be subtracted, the remainder will be the reversion of two joint lives after one.

Ex. Let the respective ages of the two expectants and possessor be 43, 54, and 66 years, and the rate of interest 4 per cent. then,

From

From the value of the two joint lives
 in expectation = 8.364
 Subtract that of the three joint lives = 5.152

The remainder = 3.212 is
 the value of the reversion of two joint lives after
 one.

PROBLEM X.

To find the value of the reversion of an annuity, which is to continue during one life, after two joint ones, at a given rate of compound interest.

72. Let xy represent the probability of the two lives in possession outliving the first year; then $1 - xy \times z = z - xy z$ will be the probability of the life in expectation outliving the two lives in possession the first year; hence,

$F - \overline{NMF}$ will be the value of the reversion; Therefore,

If, from the value of the expectant's life, the value of the three joint lives be taken, the remainder will be the value of the reversion of one life, after two joint lives.

Ex. Let the age of the life in expectation be 43 years, and the ages of the two lives in possession 54 and 66 years, allowing compound interest at 4 per cent. Then,

From the value of the expectant's life = 12.683
 Subtract the value of the three joint lives = 5.152

There remains the value of the reversion = 7.531

PROBLEM

PROBLEM XI.

To find the value of the reversion of an annuity, which is to continue during the life of a given age, after the longest of two lives, whose ages are also given, at a given rate of compound interest.

73. Let $x, y,$ and z represent the respective probabilities of the three lives continuing one year. Then $z \times \overline{1-x} \times \overline{1-y} = z - xz - yz + xyz$ will be the probability of the life in expectation outliving the other two the first year; and $F - \overline{FM} - \overline{NF} + \overline{NMF}$ will be the value of the reversion of the life in expectation after the other two.

But if the respective values of the longest of two and three lives, Prob. IV. and V. be compared with this, it will appear to be their difference; Therefore,

If, from the value of an annuity on the longest of three lives, the value of an annuity on the longest of the two in possession be taken, the remainder will be the value of the reversion on the third life, after the other two.

Ex. A, who is 43 years of age, is intitled to the reversion of an estate for his life, after the decease of his father and of his mother-in-law, whose respective ages are 66 and 54 years; sought, the value of A's interest in the estate, allowing 4 per cent.

From

From the value of the longest of the three lives	=	15.190
Subtract the value of the longest of the two possessors lives	=	11.904
Remains the value of the reversion,	=	<u>3.286</u>

PROBLEM XII.

To find the value of the reversion of an annuity, which is to continue during the longest of two lives after one, at a given rate of compound interest.

74. Let $\overline{1-x} \times \overline{1-y} = \overline{1-x-y+xy}$ represent the probability of the two lives failing in one year, which, being subtracted from unity, leaves $x+y-xy$, the probability of one, at least, of the two lives outliving the year; and this last expression, being multiplied by $\overline{1-z}$, gives $x+y-xy-xz-yz+xyz$, the probability that the longest of the two lives will survive the third the first year. Hence $N+M-\overline{NM}-\overline{NF}-\overline{MF}+\overline{NMF}$ will be the value of the reversion of an annuity upon the longest of two lives after one; but this value is the difference of the results of Prob. I. and V. Therefore,

If, from the longest of the three lives of a possessor and two expectants, be taken the value of the possessor's life, the remainder will be the reversion of the longest of two lives after one.

Ex. Let the respective ages of the two expectants be 43 and 54, and that of the possessor 66 years, and the rate 4 per cent.

From

From the value of the longest of the three lives =	15.190
Take the value of the possessor's life =	7.333
Remains the value of the reversion =	7.857

P R O B L E M XIII.

To find the value of the reversion of an annuity, where a given term of years is concerned.

75. 1st, To find the value of the reversion of an assigned life after a given term of years.

Ex. A, aged 15 years, expects to enter upon an estate of L. 100 per annum, after the expiration of 10 years, which he is to hold during life; sought, the value of his expectation, reckoning interest at 4 per cent. Here,

From the value of the proposed life =	16.410
Subtract the value of an annuity for the given term of years, on the contingency of its ceasing upon the extinction of the proposed life, found by Prob. VI. =	7.540

The remainder multiplied by 100 =	}	8.870
L. 887, is the value of A's expectation,		

2d, To find the value of the reversion of an annuity, for the remainder of a given term of years after an assigned life.

Ex. A, aged 25 years, who has the right of an annuity

annuity for 31 years certain, makes over the reversion thereof to B and his heirs, to enjoy the same after his decease for the remainder of the said term; sought, the value of B's expectation, reckoning interest at 4 per cent.

From the value of the annuity certain	
for 31 years, =	17.588
Subtract the value of the annuity for	
the same term, on the contingency of	
its failing on the extinction of A's	
life, found by Prob. VI. =	13.858
Remains the value of B's expectation, =	3.730

3d, To find the value of the reversion of an annuity for a certain number of years, after an assigned life.

Ex. Let the given age be 50, the term of years 12, and the rate of interest 4 per cent. Then,

The value of the life is 11.344, the nearest to which in Tab. V. of annuities certain is 11.118, corresponding to 15 years;

But $12 + 15 = 27 = 16.496$ the value of the life and reversion;

Subtract the value of the life = 11.344

There remains, $5.152 =$ the value of the reversion of 12 years, after a life of 50.

4th, A person, 35 years of age, wants to buy an annuity for what may happen to remain of his life after 50 years of age; sought the value of such an annuity in ready money, and also in annual payments for 15 years, till he attains to

Z the

J

the said age, subject in the mean time to failure, should his life fail.

The present value of such an annuity is equal to the value of a life at 50, multiplied by the present value of L. 1 to be received at the end of 15 years, and also by the probability that the given life will continue so long. That is,

$11.344 \times .5553 \times \frac{346}{490} = 4.44$, the number of years purchase that ought to be given for the annuity, reckoning interest at 4 per cent.

Now, supposing the annuity L. 50, its value in present money is L. 222,

Or thus: From the value of a life of 35
years, =

13.979

Subtract its value for 15 years, subject
to failure, =

9.533

Remains the years purchase of the annuity, = 4.446

And L. 222, the value of the annuity in ready money, divided by 9.53, its value for 15 years, gives L. 23.3, the annual payments till the life attains to 50 years.

5th, To find the present value of an annuity to be enjoyed by one life, for what may happen to remain of it beyond another life of equal age, after a given term; that is, provided both lives continue to the end of a given term of years. Here,

Multiply the value of the annuity for the two joint lives, plus the given term of years, discounted for the given term, by the probability that the two given lives shall both continue the given term; the product will be the answer.

Ex.

Ex. Let the two lives be each 30, the term 7 years, the annuity L. 10, and the interest 4 per cent.

From the value of a single life at $30+7 = 13.67$
 Subtract the value of two joint lives, each $37 = 10.25$
 Remains, the value of an annuity for the
 life of a person of 37 years of age, after
 another of the same age, which, being } 3.42
 discounted for 7 years, $= 2.6$.

The probability that two lives at 30 shall continue 7 years is (by Mr De Moivre's hypothesis)
 $\frac{49}{56} \times \frac{49}{56} = .765$; and $2.6 \times .765 = 1.989$, the number of years purchase which ought to be given for an annuity, to be enjoyed by a life now 30, after a life of the same age, provided both continue 7 years; and the annuity being L. 10, its present value is L. 19.89.

In like manner, supposing the term one year, and the ages and the rate of interest the same, the present value of the same reversionary annuity is L. 32.4; and if the term is 15 years, the value is L. 9.7.

Hence, supposing the scheme of a society for granting annuities to widows to be, that if a member lives 1, 7, or 15 years after admission, his widow, of the same age with himself, shall be intitled to an annuity of L. 20, L. 30, or L. 40; we may find what ought to be the annual payment of each member of a given age, and at a given rate of compound interest.

By the above, the value of L. 10 per annum, to a life of 30 after another of 30, provided the joint lives do not fail in one year; is L. 32.4, at 4 per cent,

The value of L. 20 per annum, in the same
 circumstances, is, therefore, = 64.80
 The value of L. 10, after 7 years, is = 19.89
 And of L. 10, after 15 years, is = 9.70

Their sum = 94.39

is the value of the expectation in a single present payment; which, being divided by 11.182, (the value of two joint lives at 30), gives L. 8.44, the annual payment during the joint lives. In like manner, were the ages 40 and 50, the annual payments would be L. 8.69 and L. 9.05.

P R O B L E M XIV.

To find the value of an annuity, or the reversion thereof, depending on survivorship, and the expectation of life.

76. 1st, A and B, whose respective ages are 25 and 40 years, enjoy an annuity equally betwixt them; which, after the decease of either of them, is to belong to the survivor for life; sought, the value of the right of each in that annuity, reckoning interest at 4 per cent.

From the value of A's life = 15.318
 Subtract half the value of the joint lives = 5.327

Remains, the value of the right of A = 9.991
 In like manner B's right = 7.869

2d, An annuity, after the decease of A, aged 66, is to be divided equally between B and C, whose respective ages are 43 and 54 years, during their joint

joint lives, and then is to go entirely to the last survivor for life; sought, the value of B's expectation, reckoning interest at 4 per cent.

From the value of the reversion of the life B
after the life A, found by Problem VIII. = 6.433
Subtract half the value of the reversion of the
joint lives of B and C, after the life A,
Problem IX. = 1.582

Remains, the value of B's expectation = 4.851

3d, A, aged 40, expects to come to the possession of an estate, should he survive B, aged likewise 40, and offers to give security for L. 40 per annum out of the estate at his death, provided he should get into possession: What sum ought now to be advanced to him, in consideration of such security, reckoning interest at 4 per cent.?

From the value of the perpetuity = 25.000
Subtract the value of the longest of two
lives, each 40 = 16.566

Remains, the value of the reversion of ditto = 8.434

which, multiplied by 40, gives L. 337.36, the value of the given estate, were it certainly to be enjoyed, after the extinction of the longest of two lives, both 40; but that A's life, in particular, should fail last, rather than B's, is an even chance: The true value of the reversion, therefore, is half the last value, or L. 168.68.

4th, B, aged 40, will, if he lives till the decease of A, whose age is 30, become possessed of an estate of L. 40 a-year; sought, the worth of his

his expectation in present money, reckoning interest at 4 per cent. Here assume two lives, whose common age is equal to that of the older of the two proposed lives, A and B, and find the expectations of life in A and B, viz. 28 and 23. Then,

From the value of the perpetuity =	25.000
Subtract the value of two joint lives, each	
40 =	9.826
Remains, the reversion of the two joint	
lives =	15.174

the half of which, viz. 7.587 would be the value of B's expectation, were their lives each 40: Therefore say,

As 28 : 23 :: 7.587 : 6.232, which, multiplied by 40, gives L. 249.28, the value of B's expectation.

If B's age be 30, and that of A 40 years ; then,

From the value of the perpetuity =	25.00
Subtract the value of the joint lives B and A,	
viz. 10.428 + 6.232, found above, =	16.66
Remains, the value of B's expectation =	8.34

5th, C and his heirs are intitled to an estate of L. 200 per annum, upon the decease of B, aged 40, provided B survives A, whose age is 30 years; sought, the value of their expectation in present money, reckoning interest at 4 per cent. Assume two equal lives, whose common age is that of the older of the two proposed lives, B and A ; then,

From

From the value of the perpetuity = 25.000
 Subtract that of the longest of two equal
 lives, each 40 = 16.566

Remains, the reversion of ditto = 8.434

the half of which, viz. 4.217, would be the years purchase required, if the ages of A and B were each 40 years: Therefore,

28 : 23 :: 4.217 : 3.464, the number of years purchase required, answering to L. 692.8.

If B's age had been 30, and A's 40 years, in this case,

From the value of the perpetuity = 25.000
 Subtract that of the longest of the two lives
 B and A, viz. 17.452 + 3.464, found a-
 bove = 20.916

Remains, the value of C's expectation = 4.084

Hence, in a widows scheme, when the respective ages of husband and wife are 30 and 25 years, and the provision for a family of children, where there is no widow, L. 200; we may find the present value of said provision, reckoning interest at 4 per cent.

By the above, the years purchase of the children's provision would be 3.128; but the interest of L. 200, for one year, at 4 per cent. is L. 8: Therefore $3.128 \times 8 = \text{L. } 25.024$, is the expectation of a family of children, without a widow.

P R O B L E M X V .

Of Insurance on Lives.

77. 1st, **I**F A, aged 30 years, is willing to advance money in hand, or to pay a certain sum yearly during life, to secure L. 100 to his heirs; or, in other words, to insure his life upon L. 100, allowing interest at 4 per cent. In this case, $N=14.684$, and (Problem VII. cor. 1.) $\frac{P}{n}=40 \times 100 = \text{L. } 40$, the sum payable in hand; which, divided by 14.684, gives L. 2.73, the annual payments during life.

2d, A, aged 40 years, agrees to advance a certain sum in hand, or to make annual payments during the space of 15 years, in case he lives so long, in order to secure L. 300 to him or his heirs; sought, the present value of said sum, or annual payments during these 15 years, reckoning interest at 4 per cent. Here, (Problem VI.), $N=8.817$, and $\frac{1-rN}{R}=.6224 \times 300 = \text{L. } 186.7$, the present value; which, divided by 8.817, gives L. 21.20, the annual payments during 15 years, in case A lives so long.

3d, A, aged 40 years, wants to make provision for B, aged 30, in case the latter should happen to survive: What ought the former to give in a single payment, and also in annual payments during their joint lives, to secure L. 500, payable at his death to the latter, reckoning interest at 4 per cent.?

In

In this case, the value of the joint lives of A and B = 10.428, L. 500, at 4 per cent, is equal to an estate of L. 20 per annum, and (Problem XIV. 4.) the value of B's expectation of surviving A is 8.34; therefore, $8.34 \times 20 = \text{L. } 166.8$, is the single payment, and 166.8, divided by 10.428, gives L. 16, the annual payments during the joint lives of A and B.

4th, An estate of L. 20 per annum will be lost to the heirs of a person now 40, should his life fail in 12 years: What ought he to give for the assurance of it for that term, reckoning interest at 4 per cent.? Or, What is the present value of the estate, to be entered upon at the failure of such a life, should that happen in 12 years?

From the value of an annuity certain for 12	
years =	9.385
Subtract the value of the life subject to fail-	
ure in 12 years =	8.154
Remains, the value of the estate for 12	} ———
years, upon the chance of the life's	
failing in that time.	

Again, 25, the perpetuity, $\times .6246$, the value of L. 1, discounted for 12 years $\times \frac{121}{445}$, the chance of the life's failing in that time, gives 4.246, the value of the estate after 12 years, upon the chance of the life's failing in that time; and these two values, added together, give 5.477, the value of insurance, corresponding to L. 109.54, which, divided by 8.154, the value of the life, quotes L. 13.43, the annual payments.

5th, There is an estate, which, if A, aged 7 years, happens to die before he attains to the age of 21, is, after his decease, to go to B and his heirs for ever; sought, the value of B's expectation, reckoning interest at 4 per cent.

If the reversion of the estate, after A's life had belonged to B and his heirs; then, in this case,

From the perpetuity =	25.000
Subtract the value of a life 7 years old =	16.698

There would remain B's expectation =	8.302
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But B's expectation is limited to 14 years of A's life; therefore,

From the perpetuity =	25.000
Subtract the value of a life 21 years old =	15.781

Remains, the reversion of a life of that age =	9.219
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and $9.219 \times .5775$, the value of L. 1, discounted for 14 years $\times \frac{592}{692}$, the chance of A's living 14 years, gives 4.554; which, being subtracted from the former value, leaves 3.748, the value of B's expectation.

Again, had the reversion of the estate been limited to B, aged, for instance, 30 years; in this case, $3.748 \times \frac{407}{531}$, the chance of B's living 14 years, gives 2.873, the value of B's expectation; which, multiplied by 4, the interest of L. 100 for one year, produces L. 11.6, the present value of L. 100 to be paid by B, in case the survivorship should take place.

PROBLEM XVI.

To find the Value of Successive Lives.

78. LET N and M represent the values of an annuity of L. 1, to continue n and m years, at a given rate of compound interest;

$$\left. \begin{aligned} \text{then } N &= \frac{1 - \frac{1}{R^n}}{r} \quad (47) \\ \text{and } M &= \frac{1 - \frac{1}{R^m}}{r} \end{aligned} \right\} \text{which gives } \left\{ \begin{aligned} \frac{1}{R^n} &= 1 - rN, \\ \frac{1}{R^m} &= 1 - rM. \end{aligned} \right.$$

Hence, $\frac{1}{R^{n+m}} = \overline{1 - rN} \times \overline{1 - rM} = 1 - rN - rM + r^2 NM$. Now,

let V be the value of an annuity, which is to continue $n+m$ years; then, $\frac{1}{R^{n+m}} = 1 - rV = \overline{1 - rN} \times \overline{1 - rM}$; which gives,

$V = \frac{1 - \overline{1 - rN} \times \overline{1 - rM}}{r} = N + M - rNM$, the value of an annuity to continue during two successive lives, whose single values are N and M .

Ex. 1st, A, aged 40 years, enjoys an annuity for his life; and, at his decease, has the nomination of a successor, aged 18 years, who is also to enjoy the annuity during his life; sought, the present

present value of the two successive lives, reckoning interest at 4 per cent. Here $N=13.196$, and $M=16.105$.

Hence, $N+M-rNM=29.301-8.5=20.810$, the value required.

In like manner, if there be three successive lives, whose single values are N , M , and F , then,

$V = \frac{1 - \frac{1}{1-rN} \times \frac{1}{1-rM} \times \frac{1}{1-rF}}{r}$ the value of three successive lives.

Again, if the two successive lives are of the same age, which is most likely, then $n+m=2n$, and

$\frac{1}{R^{2n}} = 1 - rV = \frac{1}{1-rN^2}$, which gives

$V = \frac{1 - \frac{1}{1-rN^2}}{r}$ the value of two equal successive lives,

and $V = \frac{1 - \frac{1}{1-rN^3}}{r}$ the value of three equal successive lives.

Ex. 2d. It is required, to find the value of four equal lives, following one another in succession, whose common age, at admission, is 24 years, reckoning interest at 4 per cent. Here $N=15.437$ and

$$\frac{1 - \frac{1}{1-rN^4}}{r} = \frac{.9786}{.04} = 24.465, \text{ the value.}$$

Universally, if N stands for the value of an annuity to continue a certain number of years then

then $\frac{1 - \frac{1}{R^n}}{r}$ will represent the value of an annuity to continue n times as long. If n is indefinite, then $1 - \frac{1}{R^n} = 0$, and $v = \frac{1}{r}$, the value of the perpetuity.

PROBLEM XVII.

Of a Copy-hold, and the renewing of Leases.

79. SUPPOSING any number of equal lives of a given age, and that upon the failing of any one, or all of them, they shall be immediately replaced, and I shall then receive a sum of money S , agreed upon; sought, the value of that expectation, and at what intervals of time I may expect to receive the said sum.

Let n denote the number of years during the intervals betwixt the payments of the sum S ; and let N represent the value of the life or lives during these intervals; then,

$$N = \frac{1 - \frac{1}{R^n}}{r} \quad (47.) \quad \text{which gives } R^n = \frac{1}{1 - rN}.$$

Now seeing a sum of money, S , is to be received, *ad infinitum*, at the equal intervals of time denoted by n ; its present value may be found from the following series, viz. $\frac{S}{R^n} + \frac{S}{R^{2n}} + \frac{S}{R^{3n}}$, &c. $= \frac{S}{R^n - 1}$ (40.) the present value of all the payments.

But

But in place of R^n , substitute its value found above, viz. $\frac{1}{1-rN}$.

then $R^n - 1 = \frac{rN}{1-rN}$, and consequently

$$\frac{S}{R^n - 1} = \frac{1-rN}{rN} \times S, \text{ the value required.}$$

Again, let $\frac{1}{1-rN} = Q$, then $R^n = Q$, and

$n = \frac{\text{Log. } Q}{\text{Log. } R}$ the intervals of time at which I may expect the sum S .

Ex. 1st. Let there be three persons, each 20 years of age, at the decease of any one of which, I shall receive L. 250, which life is to be immediately replaced, and whenever a life becomes vacant, I or my heirs are to receive a like sum; sought, the value of that expectation, and at what equal intervals of time I may expect the said sum, allowing interest at 4 per cent.

Here, $N = 9.192$ by Prob. III. and

$$\frac{1-rN}{rN} \times S = 1.72 \times 250 = \text{L. } 409, \text{ the value required.}$$

$$\text{Again, } n = \frac{.1990631}{.0170333} = 11.68 \text{ years of interval.}$$

Ex. 2d. A purchases a leasehold estate, upon three equal lives, aged 15 years, for the sum of L. 2000, on condition that his heirs fill up the lease continually, whenever all the lives become vacant, paying a fine of L. 1000; sought, the present value of the whole purchase allowed for the estate, with the annual rent answering thereto, and at what equal intervals of time the heirs of

A

A may expect to pay the said fine, allowing interest at 4 per cent.

Here $N=20.25$, by Prob. III. and

$\frac{1-rN}{rN} \times s = .2346 \times 1000 = \text{L. } 234.6$ the value of the fines;

To which add, 2000.0

Their sum is, 2234.6 the value of the purchase, whose annual rent, at 4 per cent. is L. 89.384; and

$n = \frac{.7212464}{.0170333} = 42.3$ years of interval.

If only one life is concerned, which, at the demise of the incumbent, is to be replaced with a successor, as in the case of a benefice, or of a copy-hold; in which, let N denote the value of the life, and s the fine, if there is any, to be paid on admission; then,

$\frac{1-rN}{rN} \times s =$ value of the copy-hold or right of patronage, to which add the sum s , if the purchase is made during the vacancy; then $\frac{s}{rN} =$ the value of the patronage.

Ex. 3d, Let an intrant's age be 30 years, the fine on admission L. 150; sought, the value of the patronage at 4 per cent. Here $N=14.684$, and $s=150$. Hence,

$\frac{1-rN}{rN} \times s = .7025 \times 150 = \text{L. } 105.375$, the value sought. Or, if the purchase be made during the vacancy, then,

$\frac{s}{rN} = \text{L. } 255.375$, the value of the patronage.

PROBLEM XVIII.

Of the Expectation of Life.

80. **T**HE expectation of life is that period of time, during which a person of a given age may justly expect to continue in being, and is always equal to the years purchase of an annuity of L. 1 during life, upon the supposition that money bears no interest.

Case 1st, It is required, to find the expectation of a single life of a given age, supposing the decrements of life to be equal.

In estimating the expectation of life, we must reckon from the present instant, and supposing n to be the complement of life, make $\frac{n}{n} = 1$, the

first term of the series, $\frac{n}{n} + \frac{n-1}{n} + \frac{n-2}{n}$, &c. expressing

the expectation of life; which being a series decreasing every instant, the last term thereof

viz. $\frac{n-n}{n} = 0$, and the number of terms, though in-

definite, is expressed by n ; therefore, $\frac{n}{n} \times \frac{n}{2} = \frac{n}{2}$ (8)

the sum of the series is equal to the expectation of that single life whose complement is n .

Ex. A person of 24 years of age, may expect to live $\frac{n}{2} = \frac{62}{2} = 31$ years longer.

2d, It is required, to find the expectation of two joint lives, of given ages.

Let the complement of the elder life be n , and that of the younger m ; then the sum of the series $\frac{n}{2} \times \frac{m}{m} + \frac{n-1}{2} \times \frac{m-1}{m} + \frac{n-2}{2} \times \frac{m-2}{m} (n)$ will be the expectation of the joint lives. Now, the terms of this series are the products of the two arithmetical progressions, which express the expectations of the single lives, carried on to a number of terms equal to n ; but the sum of the first is $\frac{n}{2}$, and, since the greatest term of the second is $\frac{m}{m}$, the least $\frac{m-n}{m}$, and the number of terms n , the sum thereof will be

$$\frac{n}{2} + \frac{m-n}{m} \times \frac{n}{2} = \frac{2m-n}{m} \times \frac{n}{2} = n - \frac{nn}{2m}; \quad (7, \text{Equ. 3d.})$$

hence, the sum of n terms of the series of products will be =

$$\frac{\frac{n}{2} \times n - \frac{nn}{2m} \times n}{n} + \frac{n+1 \times n \times n-1}{2 \cdot 2 \cdot 3} \times \frac{1}{n} \times \frac{1}{m}, \text{ which being}$$

reduced, becomes $\frac{n}{2} - \frac{nn}{4m} + \frac{nn-1}{12m}$, and, neglecting $\frac{1}{12m}$, it will become $\frac{n}{2} - \frac{nn}{6m}$, the expectation required.

Ex. Two persons, whose respective ages are 30 and 25 years, may expect to live together 28—8.568 = 19.43 years.

If the two ages are equal, the expectation of their joint lives will be $\frac{n}{2} - \frac{nn}{6n} = \frac{n}{2} - \frac{n}{6} = \frac{n}{3}$

3d, It is required, to find the expectation of any number of equal joint lives, of a given age.

Let there be, for instance, two equal lives;
B b
then

then the first term of the series, expressing the expectation of their joint continuance, will be $\frac{n}{n} \times \frac{n}{n} = 1^2$; and the other terms will also be squares, so that the whole will be a series of squares (whose roots are in arithmetical progression) beginning with 1, and ending in a cypher, and n the number of terms; therefore, calling m the index of the power in the several terms, $\frac{n \times 1}{m+1} = \frac{n}{3}$ will be the sum of the series, or expectation required: See the following note *. Or let there be three equal lives: then the first term of the series expressing the expectation of their joint continuance will be $\frac{n}{n} \times \frac{n}{n} \times \frac{n}{n} = 1^3$; and $\frac{n \times 1}{m+1} = \frac{n}{4}$, their expectation required; and universally calling any number of equal lives m , and their complement of life n , then $\frac{n}{m+1}$ will express the expectation of their joint continuance in being; see the following note *.

Ex.

* THE ARITHMETIC OF INFINITIES.

If there be any series of quantities, all of the same power, whose roots are in arithmetical progression, 0, 1, 2, 3, 4, &c. and let m denote the index of the power; N the number of terms; L the last

term; and S the sum of the series; then, $\frac{NL}{m+1} = S$. Thus,

$$\begin{array}{l}
 0, \quad 1, \quad 2, \quad 3, \quad 4, \\
 0^2, \quad 1^2, \quad 2^2, \quad 3^2, \quad 4^2, \\
 0^3, \quad 1^3, \quad 2^3, \quad 3^3, \quad 4^3, \\
 0^4, \quad 1^4, \quad 2^4, \quad 3^4, \quad 4^4, \\
 \sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \\
 \sqrt[3]{0}, \sqrt[3]{1}, \sqrt[3]{2}, \sqrt[3]{3},
 \end{array}
 \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \frac{1}{2} \\ \frac{1}{3} \end{array} \right]
 m = \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ \frac{1}{2} \\ \frac{1}{3} \end{array} \right]
 m+1 = \left[\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ \frac{3}{2} \\ \frac{4}{3} \end{array} \right]
 \frac{NL}{m+1} = \left[\begin{array}{c} \frac{NL}{2} \\ \frac{NL}{3} \\ \frac{NL}{4} \\ \frac{NL}{5} \\ \frac{2NL}{3} \\ \frac{3NL}{4} \end{array} \right] = S$$

This

Ex. Fifty persons, whose age at a medium is 30 years, may expect to continue all in being $\frac{56}{51} = 1$ year 36 days.

4th, It is required to find the expectation of three joint lives, whose ages are given. Let the complements of the eldest, second, and youngest, be severally denoted by n , m , and t ; then by following a process similar to that observed in case 2d, the expectation of their joint continuance would be found to be,

$$\frac{n}{2} - \frac{n^2}{6} \times \frac{1}{m} + \frac{1}{t} + \frac{n^3}{12mt}. \quad \text{See the note in case 9th.}$$

Ex. Three persons, whose respective ages are 50, 45, and 35 years, may expect to continue all in being, $18 - 9.59 + 1.86 = 10.27$ years.

5th, It is required, to find the expectation of the longest of two lives, of given ages. Here, by reasoning as in Prob. IV, and calling N and M the expectations of the single lives, and $\overline{NM}$ that of their joint continuance; then,

$N + M - \overline{NM}$ will be the expectation of the longest of these two lives.

Ex. Sought, the expectation of the longest of

This may be proven by induction, whereby it is shewn, that the greater the number of terms is, the nearer will the expression $\frac{NL}{m+1}$ approach to the sum of the series, and is only equal to it when the number of the terms is indefinite.

two lives, whose respective ages are 30 and 25 years. Here,

$28 + 30.5 - 19.43 = 39$, the expectation of the longest liver,

6th, It is required, to find the expectation of the longest of three lives of given ages. Let N , M , and F represent the expectations of the three single lives, and NMF that of their joint continuance; then, by reasoning as in Prob. V. $N + M + F - NM - NF - MF + NMF$ will be the expectation of the longest of these three lives.

7th, If the ages are equal, then the sum of the

series $1 - \frac{0}{n} + 1 - \frac{1}{n} + 1 - \frac{2}{n} + \dots$ &c. will be the expectation of the longest liver: but this series is equal to a number of units whose sum is n , minus a series of m powers, whose roots are in arithmetical progression, beginning with 1, and ending in 0; therefore $n - \frac{n}{m+1} = \frac{mn}{m+1}$ is the sum of the series. Hence,

$$\left. \begin{array}{l} 2n \\ 3 \\ 3n \\ 4 \\ 4n \\ 5 \\ mn \\ m+1 \end{array} \right| = \text{The expect. of the longest of} \left. \begin{array}{l} 2 \\ 3 \\ 4 \\ m \end{array} \right| \begin{array}{l} \text{equal lives} \\ \text{whose com-} \\ \text{plement of} \\ \text{life is } n. \end{array}$$

Ex. Let there be 100 persons, whose age, at a medium, is 30 years, some of them may expect to live $\frac{5600}{101} = 55.44$ years,

8th,

8th, It is required, to find the expectation of widowhood in married persons, whose ages are given.

$$\begin{array}{rcl} \text{From the expectation of the longest liver} & = & N + M - \overline{NM} \\ \text{Subtract that of their joint continuance} & = & \quad \quad \quad + \overline{NM} \\ \hline \end{array}$$

$$\text{Remains the expectation of widowhood} = N + M - 2\overline{NM}$$

Ex. In married persons whose respective ages are 30 and 25 years, the expectation of widowhood is $58.5 - 38.86 = 19.64$.

In married persons of equal ages, the expectation of widowhood is equal to that of marriage, seeing $\frac{2n}{3} - \frac{n}{3} = \frac{n}{3}$, which expresses either of them.

9th, If the expectation of life, for any number of years not exceeding n the complement thereof, be required; in this case calling x the number of years, the sum of the series $\frac{n}{n} + \frac{n-1}{n} \dots + \frac{n-x}{n}$ will express this expectation; but the sum of the first and last terms $= \frac{2n-x}{n}$, and x the number of terms; hence, $\frac{2n-x}{2n} \times x$ (7. Equ. 3.) is the expectation required. When $x=n$, the expression becomes $\frac{n}{2}$, and gives the expectation of the assigned life for its whole possible duration.

Ex. The expectation of life for 15 years, in a person aged 30, is $\frac{97}{112} \times 15 = 12.99$ years.

Here it may be observed, that the above expression also discovers the number of persons alive, to which one person added annually to a society, or left annually in widowhood, at a given age, will increase

increase in x years. Thus, calling $N = \frac{n}{2}$ the expectation of a widow's life; and supposing one to come upon a society every year, the number of annuitants alive, deduced from hence, will, in x years, be $\frac{4N-x}{4N} \times x$. And calling W the number of widows left annually, at the same age, then $\frac{4N-x}{4N} \times Wx$ will be the number of annuitants alive at the end of x years. The maximum, therefore, is WN , which will always happen when $x=n$, the complement of the widow's life.

Ex. Let there come yearly upon a fund, 19.3 widows, whose expectation of life is 19.64, (case 8th;) it is required, to find the number to which they will increase in 15 years; and what will be their maximum. Here,

$$x = 15, W = 19.3, N = 19.64, \text{ and,}$$

$\frac{4N-x}{4N} \times Wx = \frac{63.56}{78.56} \times 289.5 = 234.2$, the number of widows at the end of 15 years. And their maximum is $WN = 379$. In like manner, the sum of

the series $\frac{n^2}{n^2} + \frac{n-1}{n^2} \dots + \frac{n-x}{n^2} = \frac{n^3}{3n^2} - \frac{n-x}{3n^2} = x - \frac{x^2}{n} + \frac{x^3}{3n^2}$ not only expresses the expectation of two equal joint lives, for the time x , but also gives the number of marriages in being together, that will, in x years, grow out of one yearly marriage between persons of equal ages, whose complement of life is n , which expression, when $x=n$, becomes $\frac{n}{3}$, the expectation of two equal joint lives; or, calling M the number of yearly marriages, their maximum

maximum will be $\frac{nM}{3}$, at the end of n years. If the persons are of unequal ages, let the complement of the oldest life be n , and that of the youngest m , the expression will be $x - \frac{n+m \times x^2}{2nm} + \frac{x^3}{3nm}$; and the number of marriages will come to a maximum when $x = n$.

Ex. Let there be a society, wherein there are 32 yearly marriages, between persons whose respective ages are 33 and 25 years; it is required, to find the greatest possible number of married persons, as also the maximum of widows and widowers in such a society. Here, $x = n = 53$, $m = 61$ and $M = 32$; hence, $53 - 49.5 + 3.5 = 19$; which, multiplied by 32, gives 608, the greatest number of married persons; which number being equal to all the widows and widowers in life at one time, and the proportion of these to one another being as 5 to 3; hence, $8:5 :: 608:380$, the maximum of widows, and 228, the greatest number of widowers.

The EXPECTATION of LIFE discovered by FLUXIONS.

1st, Let $\frac{n-x}{n}$ represent the probability that a life, whose complement is n , will continue to the end of x time. Then $\frac{\dot{nx} - \dot{xx}}{n}$ will be the fluxion of the sum of the probabilities, whose fluent, (to find which, seeing the fluxion of x^2 is $\dot{xx} + x\dot{x} = 2x\dot{x}$, divide the whole by $\dot{x}$, add unity to the exponent of the power of x in each term, and divide by the exponent so increased), viz. $x - \frac{x^2}{2n} = \frac{2n-x}{2n} \times x$, is the sum itself for the time x ; this, when $x = n$, becomes $\frac{n}{2}$, and gives the expectation of the assigned life for its whole possible duration.

2^d, Let

Supplement to Prob. XVIII.

Let N, M, P , and T represent the respective expectations of 4 lives, whose complements are n, m, p , and t ; and c , the arithmetical mean complement

2d, Let $\frac{n-x \times m-x}{nm}$ be the probability that two unequal joint lives will continue to the end of x time; $\frac{n-x \times m-x}{nm} \times \dot{x}$, will be the fluxion of the sum of the probabilities; the fluent of this expression, viz. $x - \frac{n+m}{2nm}x^2 + \frac{x^3}{3nm}$, when $x=n$, becomes $\frac{n}{2} - \frac{n^2}{6m}$, the expectation of two unequal joint lives, whose complements are n and m .

3d, Since $\frac{n-x}{n^2}$ is the probability that two equal joint lives will continue to the end of x time, $\frac{n-x}{n^2} \times \dot{x}$ will be the fluxion of the sum of the probabilities, whose fluent, viz. $x - \frac{x^2}{n} + \frac{x^3}{3n^2}$, when $x=n$ becomes $\frac{n}{3}$ the expectation of two equal joint lives. Universally let the number of equal lives be m , then $\frac{n}{m+1}$ will be the expectation of their joint continuance.

4th, Let $\frac{n-x \times m-x \times t-x}{nmt}$ be the probability that three unequal joint lives will continue to the end of x time; then this expression, multiplied by $\dot{x}$, produces

$$\frac{-mt\dot{x}x + t\dot{x}^2x}{nmt} - \frac{nt\dot{x}x + m\dot{x}^2x - x^3\dot{x}}{nmt} - \frac{-mn\dot{x}x + n\dot{x}^2x}{nmt} = \text{the fluxion of the sum of the probabilities; the fluent of which expression, being reduced and } x \text{ put } = n,$$

becomes $\frac{n}{2} - \frac{n^2}{6} \times \frac{1}{m} + \frac{1}{t} + \frac{n^3}{12mt} = \text{the expectation of three joint lives whose respective complements are, } n, m, \text{ and } t.$

5th, Let

plement of all the lives wanting n , the complement of the oldest; then,

$\frac{1}{n}$,

5th, Let $\frac{x}{n} \times \frac{x}{n} = \frac{x^2}{n^2}$ be the probability that two equal lives, whose complement is n , will fail in the time x , then $1 - \frac{x^2}{n^2} \times x$ will be the fluxion of the sum of the probabilities that one of them at least will survive the time x ; the fluent of this expression, viz. $x - \frac{x^3}{3n^2}$, when $x = n$, becomes $n - \frac{n^3}{3} = \frac{2n^3}{3}$, the expectation of the longest liver. Universally let the number of equal lives be m , then $\frac{mn}{m+1}$ is the expectation of the longest liver.

6th, Let $\frac{n-x}{n} \times \frac{2x}{n}$ be the probability that there will be a survivor of two equal joint lives at the end of x time; then $\frac{2x}{n} - \frac{2x^2}{n^2} \times x$ is the fluxion of the sum of the probabilities, whose fluent, viz. $\frac{x^2}{n} - \frac{2x^3}{3n^2}$, when $x = n$, becomes $n - \frac{2n^3}{3} = \frac{n^3}{3}$, the expectation of survivorship between two equal lives, and is the same with that of their joint continuance.

7th, Let $1 - \frac{x}{n} \times \frac{x}{m}$ be the probability that the oldest of two lives will survive the other at the end of x time; then, $\frac{x}{m} - \frac{x^2}{nm} \times x$ will be the fluxion of the sum of the probabilities, whose fluent $\frac{x^2}{2m} - \frac{x^3}{3nm}$, when $x = n$, becomes $\frac{n^2}{6m}$. Hence the expectation of survivorship on the part of the oldest = $\frac{n^2}{6m}$.

$$\text{Add} \quad \frac{\frac{n}{2} - \frac{n}{2}}{2}$$

of the youngest

$$= \frac{\frac{n}{2} - \frac{n}{2} + \frac{n^2}{6m}}{2}$$

Add

$$+ \frac{\frac{n^2}{6m}}{6m}$$

Expectation of widowity

$$= \frac{\frac{n}{2} - \frac{n}{2} + \frac{n^2}{3m}}{2}$$

1st, The expectation of the joint continuance of 2 unequal lives $= \frac{n}{2} - \frac{n^2}{6m} = N - \frac{n}{3m} N = \overline{NM}$.

of 3 $= \frac{n}{2} - \frac{2n^2}{6c} + \frac{n^3}{12mp} = \overline{NMP}$.

of 4* $= \frac{n}{2} - \frac{3n^2}{6c} + \frac{3n^3}{12c^2} - \frac{n^4}{20mpt} = \overline{NMPT}$.

of m equal lives $= \frac{n}{m+1}$
 of m unequal lives $= \frac{\frac{n}{m+1}}{m+1} - \frac{\frac{n^{m+1}}{m+1 \times nmpt}}{m+1 \times nmpt}$ | n being the mean comple. of all the lives.

Expectation of widowity, as on p. 201. $=$

$$\frac{m}{2} - \frac{n}{2} + \frac{n^2}{3m}$$

Add

$$\frac{n}{2} - \frac{n^2}{6m}$$

_____ of the longest liver $=$

$$\frac{m}{2} + \frac{n^2}{6m}$$

8th, Let there be three joint lives, whose respective complements are n , m , and t ; the expectation of survivorship on the part of

the oldest $= \frac{n^2}{12mt}$

the second $= \frac{n^2}{12mt}$

the youngest $= \frac{t^2}{12mn}$

* To abridge the labour of calculating the expectation either of the joint continuance, or of the longest liver of m unequal lives, whose mean complement is n ;

In the fluent $\frac{x^{m+1}}{m+1 \times nmpt}$, which expresses the sum of the probabilities that all the lives will fail in the time, x ; let $x = n$, the mean complement of all the lives, and subtracting that fluent from

$\frac{2n}{m+1}$, we shall have $\frac{2n}{m+1} - \frac{\frac{n^{m+1}}{m+1 \times nmpt}}{m+1 \times nmpt} =$ expectation of their

joint continuance; and $n -$ expectation of their joint continuance $=$ that of the longest liver.

2d, The expectation of the longest liver

of 2 unequal lives $= N + M - \overline{NM} = M + \frac{n}{3m} N.$

of 3 $= \frac{n+m+p}{3} - \overline{NMP}.$

of 4 $= \frac{n+m+p+t}{4} - \overline{NMPT}.$

of m equal lives $= \frac{mn}{m+1}$

of m unequal lives $= n - \text{Expectation of their joint continuance}$ whose mean complement is n .

of q lives out of $m = \frac{qn}{m+1}$

Or in the case of 4, 5, 6... m unequal lives, find their mean age, and consider it as the age of so many equal lives.

Ex. Sought, the expectation of the joint continuance and of the longest liver of three lives, whose respective complements are 40, 50, and 60. Here, $20 - 9.77 + 1.77 = 12$, Expect. of their joint continuance. $50 - 12 = 38$, — of the longest liver.

3d, If t , the time, be sought, in which p , a part of m equal lives, will fail, whose complement is n ; here,

$\frac{pn}{m+1} = t$, the time sought.

$\frac{m+1}{n} \times t = p$; and making $t = 1$,

$\frac{m+1}{n} = p$, the number which fail in one year; hence,

In any number of lives, their expectation of life, and the number which die yearly, are reciprocal of, and discover one another.

4th, Let there be any considerable number of lives, in their natural order, extending from a given

given age, whose complement is n , to the utmost period of human life; then,

1st, $\frac{2n}{3}$ = their mean complement of life.

The given age + $\frac{n}{3}$ = the mean age; and, $\frac{n}{3}$ = their mean expectation of life.

Thus in a table of equal decrements, where the complements of life form a series of squares, $\frac{2n}{3}$ = $\frac{n^2 \times n}{3} = \frac{2n}{3}$ the mean complement, &c.

Ex. 1st, In any approved table of observations, such as that of Dr Halley, where 90 is the utmost extent of human life, 30 will be the mean age of all the persons in the table, and its expectation, viz. 28, will be that of human life, or of infancy by the table.

Moreover, $28 \times 2 + 30 = 86$, is with propriety put as the utmost extent of human life by Mr De Moivre, in his Hypothesis of equal decrements, which is adapted to Halley's table.

Ex. 2^d, The marriages of ministers wives, in the established Church, take place from the age of 18 to 45 years; hence, $\frac{45-18}{3} + 18 = 27$, the mean age of marriage, which is also the mean age of the youngest widows; hence, $\frac{86-27}{3} + 27 = 46\frac{2}{3}$, the mean age of widowity, whose expectation of life, viz. $19\frac{2}{3}$, multiplied by 19.3, the number of yearly, will produce 379.5 $\frac{2}{3}$ the maximum of annuitants,

3^d, In such a number of lives, the expectation

of the longest liver is nearly equal to the complement of life in the youngest.

4th, In such a number of lives, which are constantly the same, being kept up by the addition of new lives as the old drop off, their expectation of life is the same with that of the youngest.

5th, To find the period of doubling the number of inhabitants in any province. When the number of births, with the addition of the annual settlers, does so far exceed the burials, that their difference bears a considerable proportion to the whole number of inhabitants, and is equal, suppose, to a 36th part thereof; the number of inhabitants, if this part continues constantly the same, will be doubled in 36 years. But, as every addition to the number of inhabitants, from the births and settlers, produces a proportionably greater number of births; if we suppose the excess to increase annually in proportion to the number of inhabitants, so as to preserve the ratio of these to one another always the same; upon this supposition, the population of a province, till it doubles itself, will exactly resemble the accumulation of money at compound interest: therefore, calling this ratio of the excess of the births and settlers above the burials, to the inhabitants, $\frac{1}{r}$;

$$\left. \begin{array}{l} \frac{1}{r}, \\ \frac{r}{r+1}, \\ \frac{r^2}{r^2+1} \end{array} \right\} \begin{array}{l} \text{in the population} \\ \text{of a province, is} \\ \text{equivalent to} \end{array} \left. \begin{array}{l} r, \\ R, \end{array} \right\} \text{in compound interest.}$$

hence,

hence, by remark 2d upon compound interest,

$$\frac{L. 2}{L. r+1} = \text{the period of doubling.}$$

Ex. Let $\frac{1}{r} = \frac{1}{36}$, then $\frac{.30103}{.0119} = 25.3$ years, the period of doubling.

If M , the amount of the population P , in the time t , be sought, when the ratio of the excess of the births above the burials to the inhabitants is $\frac{1}{r}$; in this case, $P \times \frac{r+1}{r} = M$, the amount.

PROBLEM XIX.

Of the Probability of Survivorship.

Case 1st, **T**wo lives being given, to find the probability of one of them fixed upon surviving the other.

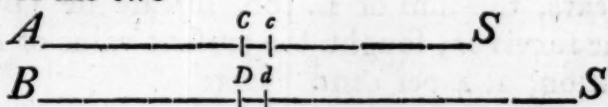
82. $A \overset{B}{|} \overset{C}{|} \overset{D}{|} \text{---} \overset{F}{|} \overset{G}{|} \text{---} S$

Let AS , the complement of life, be divided into an indefinite number of equal parts, AB , BC , CD , &c. representing moments of time; then will the probabilities of living from A to B , from A to C , from A to D , &c. be represented by the fractions $\frac{BS}{AS}$, $\frac{CS}{AS}$, $\frac{DS}{AS}$, &c. and the probability of life's failing in any interval of time AF is measured by the fraction $\frac{AF}{AS}$. Again, when the interval AF is once past, the probability of life's continuing from F to G , is $\frac{GS}{FS}$, and of its failing in that interval, is $\frac{FG}{FS}$. Hence,

The

The probability of life's continuing from A to F , and then failing from F to G , is $\frac{FS}{AS} \times \frac{FG}{FS} = \frac{FG}{AS}$; of consequence, the probabilities of life's failing in any two or more equal intervals of time betwixt A and S , are exactly the same, the estimation being made at A considered as the present time.

These things being premised, let the complements of the two



lives be $AS=n$, and $BS=m$; upon which take the two intervals AC , $BD=x$, as also the two moments Cc , $Dd=\dot{x}$; then $\frac{n-x}{n} = 1 - \frac{x}{n}$ is the probability of the first life's continuing to the end of the time x , or beyond it; and $\frac{\dot{x}}{m}$ is that of the second's continuing from B to D , and then failing in the interval Dd ; hence, $1 - \frac{x}{n} \times \frac{\dot{x}}{m} = \frac{\dot{x}}{m} - \frac{x\dot{x}}{nm}$ is the probability of these two events happening together at the end of the time x , or it is the fluxion of the sum of the probabilities, whose fluent, viz. $\frac{x}{m} - \frac{x^2}{2nm}$ is the sum itself for the time x . Let now n be put $=x$, then the probability of the first life's surviving the second will be $\frac{n}{2m}$, which subtracted from unity, leaves the probability that the second life will survive the first.

Hence the probability of survivorship on the part of the oldest life, whose complement is $n = \frac{n}{2m}$.
of the youngest, ————— is $m = 1 - \frac{n}{2m}$.
Ex.

Ex. Let $n=46$ and $m=56$; then,

$$\left. \begin{array}{l} \frac{n}{2m} = .41 \\ 1 - \frac{n}{2m} = .59 \end{array} \right\} \text{The chance of survivor. on the part of } \left\{ \begin{array}{l} \text{the oldest.} \\ \text{the youngest.} \end{array} \right.$$

When the lives are equal, $\frac{1}{2}$ is this probability on the part of either of them.

A, aged 60, whilst in health, bequeaths to *B*, aged 30 years, the sum of L. 500, in case he (*B*) shall be the survivor; sought, the present value of *B*'s expectation, at 4 per cent. Here,

$n=26$, $m=56$, and $P=15.983$. Hence,

$$500 \times \frac{P}{n} = \text{L. } 307.365$$

$$\text{multiplied by } 1 - \frac{n}{2m} = .767$$

produces *B*'s expectation = L. 235.75

Case 2d, Let there be three lives, *A*, *B*, and *C*, whose respective complements are n , m , and t ; it is required, to find the probability that any one of them fixed upon will survive the other two.

The fluent of the expression, $1 - \frac{x}{n} \times \frac{x}{m} \times \frac{x}{t}$,

being taken, viz. $\frac{x^2}{2mt} - \frac{x^3}{6nmt}$, and x put $=n$, we shall

have $\frac{n^2}{3mt}$ = the probability that the oldest life *A* will survive *B* and *C*. Again, the probability, that the second life *B* will survive the other two, is, $\frac{m}{2t} - \frac{n^2}{6mt}$; and these two, subtracted from unity, leave the probability of the youngest life *C* surviving *A* and *B*. Hence the probability of survivorship on the part of

$$\begin{aligned} A, \text{ after } B \text{ and } C &= \frac{n^2}{3mt} \\ B, \text{ — } A \text{ and } C &= \frac{m}{2t} - \frac{n^2}{6mt} \\ C, \text{ — } A \text{ and } B &= 1 - \frac{m}{2t} - \frac{n^2}{6mt} \end{aligned}$$

Ex. Let $n=40$, $m=50$, and $t=60$. Then,

$$\left. \begin{aligned} .17\frac{1}{2} &= A's \\ .32\frac{1}{2} &= B's \\ .49\frac{1}{2} &= C's \end{aligned} \right\} \text{ chance of surviving } \left\{ \begin{aligned} B \text{ and } C. \\ A \text{ and } C. \\ A \text{ and } B. \end{aligned} \right.$$

When the lives are equal, $\frac{1}{3}$ will be the probability that any one of them fixed upon will survive the other two.

Case 3d, Let there be three lives, A, B, and C. It is required, to find the probability of survivorship on the part of any two of them after the third.

The fluent of the expression $1 - \frac{x}{n} \times 1 - \frac{x}{m} \times \frac{x}{t}$, being taken, and x put $= n$, will give the probability that the two oldest will survive the youngest, $= \frac{n}{2t} - \frac{n^2}{6mt}$. Hence the probability of survivorship on the part of

$$\begin{aligned} A \text{ and } B \text{ after } C &= \frac{n}{2t} - \frac{n^2}{6mt} \\ A \text{ and } C \text{ — } B &= \frac{n}{2m} - \frac{n^2}{6mt} \\ B \text{ and } C \text{ — } A &= 1 - \frac{n}{2m} - \frac{n}{2t} + \frac{n^2}{3mt} \end{aligned}$$

Ex. Let $n=40$, $m=50$, and $t=60$. Then,

$$\left. \begin{aligned} .24\frac{1}{2} &= A \text{ and } B's \\ .31\frac{1}{2} &= A \text{ and } C's \\ .44\frac{1}{2} &= B \text{ and } C's \end{aligned} \right\} \text{ chance of surviving } \left\{ \begin{aligned} C. \\ B. \\ A. \end{aligned} \right.$$

When the lives are equal, $\frac{1}{3}$ will be the probability

lity that any two of them fixed upon will survive the third.

Case 4th, It is required, to find the order of survivorship betwixt the three lives, A, B, and C, whose respective complements are n , m , and t ; or the probability that A shall survive B, and B survive C, &c.

$$\begin{array}{lcl}
 1^{st}, & A, B, C, & = \frac{n^2}{6mt} \\
 2^{d}, & A, C, B, & = \frac{n^2}{6mt} \\
 3^{d}, & B, A, C, & = \frac{n}{2t} - \frac{n^2}{3mt} \\
 4^{th}, & B, C, A, & = \frac{m}{2t} - \frac{n}{2t} - \frac{n^2}{6mt} \\
 5^{th}, & C, A, B, & = \frac{n}{2m} - \frac{n^2}{3mt} \\
 6^{th}, & C, B, A, & = 1 - \frac{m}{2t} - \frac{n}{2m} + \frac{n^2}{6mt}
 \end{array}$$

Here it is manifest, that these six orders, taken two and two, such as the 1st and 2^d, or the 1st and 3^d, are respectively equal to the first equations in the two foregoing cases, &c.

Ex. Let $n=40$, $m=50$, and $t=60$. Then,

Order 1 st , = .088	Order 4 th , = .177
2 ^d , = .088	5 th , = .227
3 ^d , = .158	6 th , = .277

When the lives are equal, $\frac{1}{3}$ will express the probability of survivorship betwixt any order of the three fixed upon.

A, aged 70 years, whilst in health, makes a will, whereby he bequeaths L. 300 to B, aged 45, and L. 500 to C, aged 36 years, with this condition, that if either of them die before him, the whole is to go to the survivor of the two: Sought, the values of the expectations of B and C, estimated

ted from the time that the will was writ, and reckoning interest at 5 per cent. Here, $n=16$, $m=41$, $t=50$, and P , corresponding to n , $=10.838$.

Was it certain that A would die before either of them,

B's expectation would be $300 \times \frac{P}{n} = \text{L. } 203.21$

C's $500 \times \frac{P}{n} = \text{L. } 338.69$

Which expectation of C, multiplied by the probability that he and B shall survive A, $= .6865$

Produces C's expectation, arising from the prospect of A's dying before either of them, $= \text{L. } 216.916$

But C hath a further expectation, arising from the prospect that he shall survive A, and that A shall survive B, in which case he shall obtain

203.21

Which expectation, multiplied by the 5th order, $= .1535$

Produces his chance of B's legacy, $= 31.193$

To which add 216.916

Their sum is the whole of C's exp. $= \text{L. } 248.11$

In like manner, B's expectation, arising from the prospect that they will both survive A, $= \text{L. } 139.504$

To which add his chance of C's legacy, $= 40.101$

Their sum is the whole of B's expectation, $= \text{L. } 179.605$

PROBLEM XX.

Of Half-yearly and Quarterly Payments of an Annuity.

83. 1st, **I**N case of an annuity $\frac{1}{2}a$, paid every half-year, during n years, at a given rate, let P be the value of the annuity a for n years, V , the value of the half-yearly annuity for the same number of years, and $d = R^{\frac{1}{2}} - 1$, the interest of L. 1 for half a year, (Table in 46th); then

$$P = \frac{a - R^n}{r}, \quad (47.) \quad V = \frac{\frac{1}{2}a - R^n}{d}, \quad (49.) \quad \text{divide both by } a - \frac{a}{R^n}, \text{ hence } P : V :: \frac{1}{r} : \frac{1}{2d}, \text{ which gives } \frac{rP}{2d} = V.$$

If quarterly, $d = R^{\frac{1}{4}} - 1$, and $\frac{rP}{4d} = V$.

Ex. Let there be an annuity of L. 30 per annum, paid every half-year, by equal portions, for the space of 20 years; sought, the value thereof, at 4 per cent. Here $\frac{1}{2}a = 15$, $P = \text{L. } 407.7$, and $d = .01984$. Hence $\frac{rP}{2d} = \text{L. } 410.96 = V$, the value sought.

Or thus: The annuitant, in case of half-yearly payments, has the advantage above the annual payments of the interest of $\frac{1}{2}a$ for half a year, that is, of $\frac{1}{4}a$ yearly, during the continuance of the annuity; which interest, at 4 or 5 per cent. is equal to $\frac{1}{100}$ or $\frac{1}{80}$ part of the annuity; therefore to the value of the annuity paid yearly, add its 99th or 79th part, the sums will be the values of the half-years payments.

2d, If an annuity is paid for life, in this case, half-yearly payments are $\frac{1}{4}$ of a year's purchase better than yearly ones.

The annuitant, by the last, hath the advantage of the interest of $\frac{1}{4}$ of the annuity during life, besides an equal chance of receiving one half-year's value more than if he had been paid yearly; which chance is therefore, at his death, worth $\frac{1}{4}$ part of the yearly annuity. Now these two advantages, it is manifest, will always amount to $\frac{1}{4}$ of a year's purchase in present value.

Again, half-yearly payments, which begin immediately, are half a year's purchase better than those that begin at the end of half a year; that is, by the above, they are $\frac{3}{4}$ of a year's purchase better than yearly payments which begin at the end of the year. But yearly payments which begin immediately are a whole year's purchase better than the same payments to begin at the end of the year: therefore the difference of value between yearly and half-yearly payments, supposing both to begin immediately, is $\frac{1}{4}$ of a year's purchase in favour of the former.

3d, To find how many years possession will reimburse an annuitant for the price paid for his annuity.

Calling n the number of years required, a purchaser will be reimbursed for his money, when the years-purchase of his annuity for n years certain is equal to the value of his life; that is, when

$$\frac{1 - \frac{1}{R^n}}{r} = N; \text{ hence } R^n = \frac{1}{1 - rN}, \text{ and } n = \frac{L \cdot \frac{1}{1 - rN}}{L \cdot R}$$

which is always less than the expectation of life considered

considered as so many years certain. (Construction of Table II.)

Ex. A person, aged 30 years, purchases an annuity for life: Sought, how long he must live before he can be reimbursed for the purchase-money, at 4 per cent.

Here $N = 14.684$, and $\frac{L. \frac{1}{1-rN}}{L. R} = \frac{.395}{.017} = 23.23$ years.

When an annuity is bought at a number of years purchase, $= \frac{1}{2r}$, in this case, $\frac{1}{1-rN} = 2$, and $n = \frac{L. 2}{L. R}$ the time in which money doubles itself at a given rate.

When the number of years purchase $= \frac{1}{r}$, n becomes indefinite, and the possession must be a perpetuity.

PROBLEM XXI.

Of the British State Lottery.

84. The State-Lottery is a species of gambling permitted by the legislature, for the purpose of securing a certain yearly revenue to government.

The Lottery anno 1791 contains 50,000 tickets; and the number and value of the prizes, begun drawing February 20. 1792, are represented in the following scheme.

SCHEME

SCHEME of British State Lottery 1791.

Number.	Prizes.	Sum.
2 of	30,000	= 60,000
2 of	20,000	= 40,000
3 of	10,000	= 30,000
5 of	5,000	= 25,000
10 of	2,000	= 20,000
15 of	1,000	= 15,000
30 of	500	= 15,000
50 of	100	= 5,000
100 of	50	= 5,000
14,150 of	20	= 283,000
1st & last 2 of	1,000	= 2,000

$$14,369 = \left\{ \begin{array}{l} \text{N}^{\circ} \text{ of} \\ \text{Prizes.} \end{array} \right\} \text{L. } 500,000 = \left\{ \begin{array}{l} \text{Sum of} \\ \text{the Prizes.} \end{array} \right.$$

In this Scheme let

$$\begin{aligned} a &= 14,369 \text{ represent the number of Prizes.} \\ b &= 35,631 \text{ ————— Blanks.} \\ a+b &= 50,000 \text{ ————— Tickets.} \\ s &= \text{L. } 500,000 \text{ ————— the sum of the Prizes.} \end{aligned}$$

From these data the following particulars are inferred.

1st, To find the value of each Ticket, &c.

$$\frac{s}{a+b} = \text{L. } 10 = v, \text{ the value of each Ticket.}$$

$$\frac{s}{a} = \text{L. } 34.79 = V, \text{ ————— Prize.}$$

put $p = \text{L. } 16.8$, the market-price of each Ticket,
 then $\overline{p-v} \times \overline{a+b} = \text{L. } 340,000$, = equal Profit of
 government, including the expence of manage-
 ment, &c.

2d, To

2d, To find the chance or probability of drawing a Prize or a Blank. Here,

Seeing $a+b : a :: 1 : \frac{a}{a+b}$. Hence the following ratios.

1st, $\frac{a}{a+b} = \frac{1}{3.48} = 1 : 3.48 -$ } the probability { a Prize.
 2d, $\frac{b}{a+b} = \frac{1}{1.4} = 1 : 1.4 +$ } of 1 Ticket { a Blank.
 3d, $\frac{a}{b} = \frac{1}{2.48} = 1 : 2.48 -$; that is, to each Prize there are 2.48 — Blanks.

These ratios may be called the indices of the Lottery, which, being pointed to, or multiplied by any number of tickets, show the probable number of prizes or blanks in these tickets.

Ex. $\frac{1}{3.48} \times 3.48 = 1 : 1$, and shows that 3.48 tickets have an equal chance of drawing 1 prize and 2.48 blanks; but the market-price of the tickets being L. 58.36, there is therefore a loss of 58.36 — $V =$ L. 23.57.

Again, $\frac{1}{1.4} \times 1.4 = 1 : 1$, and shows that 1.4 tickets have an equal chance of drawing 1 blank, and .4 of a prize: there is therefore a loss of 23.52 — 13.92 = L. 9.6.

3d, To find how many Blanks or Prizes there are in n Tickets.

In n tickets { $\frac{n}{3.48}$ prizes, and $n - \frac{n}{3.48}$ blanks.
 there are { $\frac{n}{1.4}$ blanks, and $n - \frac{n}{1.4}$ prizes.

Ex. Let $n = 20$, there will be $\frac{20}{3.48} = 5.75$ prizes,
 and

and $20 - 5.75 = 14.25$ blanks. At the same time, the greater the number of tickets, the greater is the probability of obtaining the proportional number of prizes: Thus, 3.48 tickets have only an equal chance of drawing 1 prize; whereas $a+b$ tickets are certain of drawing a prizes: therefore the chance of drawing a proportional number of prizes varies from probability to certainty in proportion to the number of tickets.

4th, To find the probability which n tickets have of drawing a certain number of prizes, not exceeding n , let p = the prizes expected in n tickets; then $\frac{1}{p} \times \frac{n}{3.48}$ = the probability required.

Ex. Let $n=5$, and $p=3$; then $\frac{5}{10.44} = \frac{1}{2.09} = 1:2$ = the probability of 5 tickets drawing 3 prizes.

5th, To find the probability which n tickets have of drawing any one particular prize in the scheme.

That 1 ticket will draw a prize

of L. 20,	} the probability =	$\frac{14.150}{a+b} = \frac{1}{3.53}$
better than L. 20,		$\frac{219}{a+b} = \frac{1}{228.3}$
of L. 1000 and upwards,		$\frac{39}{a+b} = \frac{1}{1282}$

That n tickets will draw a prize

better than L. 20,	} the probability =	$\frac{n}{228.3}$
of L. 20,000,		$\frac{n}{25,000}$

6th, To find the probability which any 2 tickets fixed upon have of drawing 2 capital prizes, for instance, of L. 10,000 each. Here, $\frac{3}{50,000} = \frac{1}{16666.6} =$ probability that 1 ticket fixed upon, will draw a prize of L. 10,000. But in the present case of 2 tickets drawing equal prizes, as these events are independent of one another, therefore $\frac{1}{16,666.6} \times \frac{1}{16,666.6} = 1 : 277,777,777.1 =$ the probability required. Hence the vast disadvantage attending shares of tickets drawing capital prizes. Thus the probability that 2 half tickets will draw the half of two L. 10,000 prizes, is as 1 : 277, &c. as above.

7th, To find the number of prizes and blanks drawn daily, during the continuance of the lottery, being 38 lawful days.

Each law-	{	Prizes =	378,	with	5 intersperfed,
ful day the		Blanks =	937,	—	25
N ^o of		Tickets =	1315,	—	30
N ^o of L. 20		Prizes =	372,	—	14

8th, Insurance in the Lottery is not only expressly prohibited by acts of parliament, on account of the bad effects it may have upon the morals and fortunes of adventurers; but it is also, upon a fair calculation, so high, that it is not to be supposed any person will risk incurring the legal penalty by such a practice. Thus,

1st, Should an adventurer insure, upon his ticket, the sum *P*, not exceeding the least prize, upon condition that the office shall make good that sum,

sum, in case of a blank, while he reserves to himself the chance of drawing a prize; fought, the premium of insurance.

It is evident, that here only the blanks need to be insured; therefore $P \times b$ will be the insurance upon the whole, $a+b$ number of tickets, and $\frac{P \times b}{a+b}$
 $= P \times \frac{1}{1.4}$ = the insurance upon each ticket.

Ex. Let $P = L. 16.8$, then $\frac{16.8}{1.4} = L. 12$ premium of insurance.

Let $P = L. 20$, the least prize, whilst the holder retains the chance of a higher one, then, $\frac{20}{1.4} = L. 14.285$, the premium of insurance.

Hence any sum P , not exceeding the least prize, multiplied by the chance of drawing a blank, in any lottery, will give the premium of insurance upon that sum.

2d, Should an adventurer insure upon his ticket the sum of $L. 50$, whilst he retains the chance of a higher prize; fought, the premium of insurance. Here $50b +$ the number of $L. 20$ prizes multiplied by $30 =$ the insurance upon the whole number of tickets; therefore,

$$\frac{50b + 14150 \times 30}{a+b} = L. 44.12, \text{ the premium fought.}$$

REMARKS.

1st, In such a Lottery, the true value of each ticket, exclusive of the expence of management, is L. 10; and it is owing to the confidence which the credulity of the public hath in the powers of fortune, that the market-price rises to 16 guineas or upwards.

2^d, The scheme of this Lottery is constructed with great art: There is a valuable bait of L. 100,000 in four prizes, hung out to catch the attention of the public; and the number of L. 20 prizes being so great, there are not 2.5 blanks to 1 prize; whereas the chance of obtaining a L. 20 prize being scarce distinguishable from that of obtaining a prize at all, the number of the former ought in all justice to be diminished, and the 2 prizes of L. 30,000 each rejected, so as to increase the number of the intermediate prizes.

PROBLEM XXII.

Of a Tontine.

85. **A** Tontine is a kind of lottery on lives, where the capital, divided into a number of shares, is formed by the voluntary subscriptions of those who chuse to risk their money, in such a scheme, upon the continuance of their lives. There are several particulars respecting a tontine which need to be determined.

1st,

1st, In a tontine, where 100 persons, for instance, whose mean age is 20 years, subscribe each of them L. 100, forming a capital of L. 10,000, which is sunk to the heirs of the subscribers, it is required to find the number of survivors at the end of 10, 20, 30, &c. years, and what premium will then be due to each survivor, reckoning interest at 5 per cent. Here, $n=66$, $m=100$, and $t=10$; hence, (supplement to problem XVIII.)

$$\frac{m+1 \times t}{n} = \frac{1010}{66} = 15.3, \text{ who fail in 10 years; but the}$$

interest of the capital being L. 500, $\frac{500}{84.7} = \text{L. } 5.9$, the interest due to each survivor at the end of that period; hence,

living 84.7, per cent 5.90, at the end of 10 years.

————— 69.4, ————— 7.20, ————— 20.

————— 54.1, ————— 9.24, ————— 30, &c.

2d, In a tontine, where the shares are in proportion to the value of life, with a view to make a dividend of the capital when the survivors can receive each of them L. 1000; in this case, if the share of a subscriber aged 10 or 12 years be L. 100, that of a child one year old will be L. 79, reckoning the value of life at 4 per cent; thus $16.88 : 13.36 :: 100 : 79$. In like manner,

from 1 to 5 the shares are from 79 to 95

—— 5 to 10 ————— 95 to 100

—— 10 to 18 ————— 100 to 95

—— 18 to 27 ————— 95 to 90

—— 27 to 39 ————— 90 to 79

In such a tontine, let there be 100 subscribers whose mean age is 20 years, and the shares at a medium L. 90 each, forming a capital of L. 9,000; it is required, to find the time in which they

they will be so far reduced, that each survivor can receive L. 1000. Here, $m=100$, $n=66$, and $p=91$. Hence,

$$\frac{np}{m+1} = \frac{6006}{101} = 59.46 = t, \text{ the time sought.}$$

3d, In a tontine to continue a limited time, it is required, to find the capital of each survivor having a single share, at the expiration of that period.

Ex. In the equitable and universal tontine, Bristol, commencing on the 16th day of January 1792, and continuing 7 years, wherein each share is 6 s. 6 d. per quarter, with 5 d. per quarter for agency, it is required to find,

1st, Out of 100 subscribers, whose mean age is 25 years, the number of lives failing in 7 years, Here $m=100$, $n=61$, and $t=7$; hence,

$$\frac{m+1 \times t}{n} = \frac{707}{61} = 11.6, \text{ the number required.}$$

2d, The amount of each share at the end of 7 years, reckoning interest at 4 per cent. Here, $a=.325$, $R^t-1=.3159$, and $r=.00985$; hence, $a \times \frac{R^t-1}{r} = \text{L. } 10.425$, the amount of each share.

But it is evident, that, as the lives fail from year to year, the 11.6 dying out of each 100 subscribers will only have contributed the half of their quota, viz. $11.6 \times 5.2125 = \text{L. } 60.465$, to be divided among 88.4 survivors; hence,

$\frac{60.465}{88.4} = .684$, to which add L. 10.425, their sum, viz. L. 11.11, is the capital of each survivor, having a single share, at the end of $7\frac{1}{2}$ years.

These quarterly payments, viz. 6 s. 11 d. laid out

out at 4 per cent. simple interest, where there is no risk, would, at the end of $7\frac{1}{2}$ years, amount to L. 11.42.

SUPPLEMENT TO CHAP. VIII.

Of Insurance from Fire.

86. **I**N the city of Edinburgh and its environs there are about 50,000 inhabitants; and of consequence, by allowing 5 persons to each family, there will be 10,000 families in that city and its neighbourhood: Now supposing that L. 100 was insured upon each of these at 2 s. 6 d. per annum, this, for each 1000 families, would produce an annual revenue to the insurance-office of L. 125.

Again, admit that 1 family out of 1000, or 10 out of the whole city, were exposed to the accident of suffering loss by fire yearly, which is a high proportion, and, we trust, seldom verified by experience; this would occasion an annual loss to the office of L. 100 upon each 1000 families; of consequence there would be a saving of L. 25 yearly, besides interest, to defray the expence of management, upon each 1000 policies at 2 s. 6 d. a-piece. Were the risks in large and small properties the same, being in proportion to their values, their policies would be equal; but from the number of hands, fires, and lights, &c. employed, and from the connection of the parts of a building, the risk attending L. 1000 capital is greater than 10 times that attending L. 100 value. Therefore, to find this risk when the policy is, for instance, 4 s. = L. .2 per cent.; here, as

INSURANCE FROM FIRE.

$$\frac{.125 \times 1000}{.2} = 625, \text{ the risk is as 1 to 625.}$$

And when the policy is 4 s. = L. .2 per cent. upon
625 capitals of L. 500 each;

$$\begin{aligned} \text{The annual revenue to the office} &= 5 \times .2 \times 625 = \text{L. } 625 \\ \text{The annual loss would be} &= 500 \end{aligned}$$

$$\text{And the yearly advantage} = \text{L. } 125$$

In like manner, when

The policy is 2 s. 6d. = .125

$$3 \text{ s.} = .15$$

$$4 \text{ s.} = .2$$

$$5 \text{ s.} = .25$$

$$6 \text{ s.} = .3$$

the risk is as 1 to

1000

833.3

625

500

416.6

$$\text{The number of policies} = 3375$$

Policy.	Capital.	M. Revenue.	Loss.	Gain.
2s. 6 d.	L. 100,	125	100	L. 25
3 s.	from L. 100 to L. 500,	312.5	250	61.5
4 s.	from L. 500 to L. 1000,	875	700	175
5 s.	from L. 1000 to L. 2000,	1875	1500	375
6 s.	L. 2000 and upwards.	2500	2000	500

5687.5

1136.5

$$\text{Add the interest upon the mean revenue at 4 per cent.} = 237.5$$

$$\text{Hence 3375 policies produce of yearly gain} = \text{L. } 1374$$

OF THE WIDOWS SCHEME

CH A P. IX.

Of the Widows Scheme in the Church of Scotland.

87. **T**HE fund established by law, for a provision to the widows and children of the ministers of the established church, and of the professors in the universities, of Scotland, is a noble monument of the knowledge, moderation, and perseverance of the late Reverend Dr Alexander Webster, one of the ministers of Edinburgh, the principal contriver and promoter of this scheme; at the same time it is a striking evidence of the experience which the ministers of the church then had, of the straitened circumstances of many widows and orphans of ministers, who, from a competency, were reduced, at once, to a state of indigence; seeing that, out of 962 ministers and professors, only 135 declined contributing to this fund; a fund, by which, when it is complete, above 370 widows will be supported, and a sum not less than L. 1360 distributed yearly among the orphans of contributors.

Previous to the establishment of the scheme, several attempts had been made, by the different synods in Scotland, to provide for the widows and orphans of ministers; all which proving ineffectual, from their limited nature, and for want of a common rule and proper authority to enforce it, Dr Webster and others, about the year 1742, prepared the plan of a scheme, which should comprehend the whole ministers of the church, and be enforced by the authority of law. For this purpose they had proposed a set of queries to the different presbyteries; and from the lists transmitted by them, anno 1742, in answer to these queries, founded upon an inquiry for 20 years back, the following facts were discovered:

1st, That the number of benefices in the church, independent of the offices in the four universities, was 970.

2^d, That, taking one year with another, 27 ministers died yearly. That 18 of them left widows, whose mean age was supposed to be 52 years; 5 of them children without a widow; and 4 of them neither widows nor children: Hence the expectation of a widow's life, by Dr Halley's table, viz. 16.66, multiplied by 18, produces 300, the greatest number of ministers widows supposed to be alive at one time.

3^d, That 3 families of children, under 16 years of age, were left annually by widows of ministers who died or married.

These facts being discovered, the salaries of the general collector and clerk were estimated at
L. 200

L. 200 yearly, which, together with a sum not exceeding L. 39 allotted to defray the incidental expences of the trustees, made the annual expence of management, in all L. 239; which afterwards, upon admission of the professors in the four universities to the privilege of the scheme, was increased to L. 250. And in order to depress the annual rates as much as possible, so as to leave room for taxes and deductions, a high rate of interest was assumed in the calculation of the rates, namely $4\frac{1}{2}$ per cent. Upon these principles and data, a scheme was prepared and laid before the General Assembly 1743, which approved thereof; and the parliament, upon the humble application of the Assembly, were pleased to establish the same by law, and enacted that the scheme should commence on the 25th day of March 1744, making the first year thereof, which ended on the 22d day of November thereafter, to consist of 8 months wanting 3 days. From the lists transmitted by presbyteries at Candlemas 1745, it appeared that the number of benefices and offices in the church and universities, namely 1011; and consequently the annual produce for the support of the fund, was considerably less, and that the number of widows to be provided for, viz. 364, was considerably greater than had been supposed in the calculation of the rates previous to the aforesaid act of parliament; and upon instituting a new calculation, founded upon Dr Halley's Table of Observations, it further appeared, that by reason of these disadvantages, the free stock would become stationary against the 1771, and that L. 10,000 would then be wanting to raise the necessary capital of L. 80,000. The trustees, for preventing this and other disagreeable consequences,

ces, did, pursuant to the order of the General Assembly 1748, petition the parliament to order certain sums, out of the first and readiest of the annual produce, to be applied from time to time towards raising the capital; and to authorise deductions, in certain cases, from the annuities of widows and the provisions of children. The parliament were pleased to enact accordingly.

During the space of 30 years, from the year 1748, the affairs of the scheme continued, as they still continue, to prosper. The trustees, however, about the year 1773, with the advice of Dr Webster, proposed a number of alterations and improvements respecting the scheme to the General Assembly. In consequence of which, after taking the sense of the church upon these alterations, the Assembly 1778 did again petition parliament to confirm the same by law; the parliament were pleased to comply with their request, by a new statute, 19 Geo. III. cap. 20.

The several particulars respecting the persons who are liable in payment to, or receive benefit from the fund, together with the terms of payment, the rise of the capital, till it amounts to L. 100,000, and the duty of the trustees, who are appointed to superintend the whole, are all accurately expressed in the late act of parliament, which is in force from the 29th day of September 1778, a copy of which is supposed to be in the possession of every contributor. In the present state of the fund, in the year 1790, it is proposed to investigate the 12 following articles:

ART. I. To find the connection betwixt the several classes of annual rates and their corresponding annuities in the widows scheme.

88. According to lists transmitted by presbyteries previous to the commencement of the scheme, the benefices in the church appeared to be, as above, 970; ministers were supposed to be admitted to their benefices at the mean age of 27 years, which, by Dr Halley's table, makes their expectation of life 30 years. The widows of ministers were supposed to live, at a medium, about 20 years; and the expence of management was estimated at L. 239, which being divided by 970, quotes .2464 l. for each contributor's annual share of expence. Now supposing an annuity of L. 5 was granted to each widow, or 10 years purchase of her annuity to a minister's children where there is no widow, it is required to find the annual rate which each contributor ought to pay, reckoning interest at $4\frac{1}{2}$ per cent. Here,

The present value of L. 1 an-

nunity for 20 years, = 13.00793

Multiplied by the annuity, = 65.03965

Produces, 65.03965, which being divided by 61.00707, the amount of L. 1 annuity for 30 years quotes, 1.0661

To which add the yearly expence of management,

.2464

Their sum is, L. 1.3125 = L. 1 : 6 : 3, the annual rate of each contributor, including his expence of management, and supposing the annuity to be L. 5. Hence, by doubling, &c. the annual rate and annuity, when

L.	s.	d.	L.	L.
the rate is	2	12	6,	the annuity = 10, and 10 years purchase = 100,
	3	18	9,	150,
	5	5	0,	200,
	6	11	3,	250.
				From

From the construction of these four classes of annual rates, it appears that, besides bearing the burden of its corresponding annuity and yearly expence of management, the first class is taxed with 5s. the second with 10s. the third, with 15s. and the fourth, with 20s. of annual expence nearly.

These four classes of annual rates are admirable for the simplicity of their construction; and being built upon the principles of the expectation of life and a high rate of interest, are as moderate as possible, and leave full room for taxes which are just, and deductions which are not oppressive, in order to their answering the purposes intended by the act of parliament. Thus,

1st, The rates themselves are taxed as above, and the higher the rate the greater is the tax.

2^d, Contributors at their marriage, and married intrants at their admission, are taxed with the payment of one year's rate of the class to which they are subjected.

3^d, The half-years rates payable out of anns and vacant stipends may also be considered as a tax, seeing the decrement of life in the incumbent is then suspended; besides the advantage which the fund receives from $\frac{4}{30}$ of the contributors who die yearly, leaving neither a widow nor children, and from the expectation of a minister's life being $2\frac{1}{2}$ years greater than is supposed in the calculation of the rates. Again,

4th,

4th, There is a deduction of one years purchase out of the provision of a widow, seeing her annuity begins to run from a full year after the last payment of her husband.

5th, There is a deduction of two years purchase out of the provision of children, which is so much less than that of a widow.

6th, There is also a deduction out of the provisions of the widows and children of those contributors who have not paid annual rates to the amount of three years of their widows annuity, without reckoning interest thereon, till the balance is recovered; besides the advantage which the fund derives from widows, not having children under 16 years of age, who happen to marry not-contributors, and of consequence are struck off from the benefit of the fund: Whereas the only advantage which the scheme affords above the calculation, is the reversion of 10 years annuity to the children under 16 years of age, of those widows who happen to die before they have completed the 11th year of their widowhood.—These rates, taxes, and deductions, together with the provision in the act of parliament to secure the rise of the capital to L. 100,000, by the application of L. 200 yearly for this purpose, will, we trust, give a stability to the scheme, and make it answer all the purposes for which it was intended.

In the construction of these rates, supposing the number of benefices and offices in the church and universities to be 1013, including the third charge in Perth and the second in Linlithgow, and fixing the annual expence of management, agreeable

greeable to the two first acts of parliament, at L. 250, we shall have $\frac{350}{1013} = \text{L. } .2467$ for each contributor's share of expence, which, for an annuity of L. 5, will also produce a rate, as above, of L. 1 : 6 : 3, nearly exact.

Had these rates been calculated upon the strict principles of reversionary annuities, the calculation would have been more laborious, and the rates and marriage-taxes much higher than they are at present in the widows scheme. Thus :

Suppose a contributor, aged 30, marries a wife of 25 years of age, whose annuity is L. 20; sought, the annual rate which he ought to pay, reckoning interest at 4 per cent. Here,

$\frac{3.85}{14.684} \times 20 + .2464 \times 4 = \text{L. } 6.23$, the annual rate required.

Again, If the same contributor becomes a widower, and, at the age of 40, marries a second wife aged 30 years, his marriage-tax would be L. 8.68. See Prob. VIII. Of a Widows Scheme.

These annual rates are due to the fund, either for a whole or an half year, at the same terms that an incumbent hath a right to his stipend or salary for the same periods, and are payable to the collector at the first term of Candlemas thereafter; excepting in the case of the half years rates payable out of anns and vacant stipends, the former of which is always half of the contributor's rate, and the latter is fixed at L. 3, 2 s. for each half-year of vacancy, these, though due at the same terms with the former, are not payable to the collector till the first term of Candlemas which shall happen a full year after the anns and vacant stipends shall be due and payable. The medium

of

of annual payments by contributors being within a trifle of the third class of L. 5, 5 s. this third class is esteemed, in all calculations respecting the scheme, as the medium of the annual rates of all the contributors, and L. 20 the medium of all the widows annuities.

ART. 2. To find the number of benefices in the church, and offices in the universities, which are all subject to the payment of one or other of the annual rates.

89. By the lists transmitted to the trustees, by presbyteries and universities, it appeared that, at Martinmas 1744, there were,

Of Benefices in the Church,	942	} 1011 in all.
And of Offices in the Universities,	69	

Of the said Benefices and Offices,	{ full 962 vacant 49 }	} 1011 in all.
------------------------------------	---------------------------	----------------

From the best information which I have received, it appears that, at Martinmas 1789, there were,

Of Benefices in the Church,	943	} 1011 in all, as above.
And of Offices in the Universities,	68	

It being left optional, by the first act of parliament, to the ministers of the church then living, to accede to, or decline the privilege of, the scheme; there were, at Martinmas 1744, 135 decliners, that is, persons who declined accepting the benefit of the scheme, consequently there remained only 827 original contributors; and as not only their successors, but the successors also of the 135 not-contributors are obliged to become subject to the scheme, the greatest number of contributors alive at one time will be 962, making

G g in

in all, with the addition of 49 vacancies, 1011 benefices in the church and universities liable to the payment of one or other of the annual rates. Hence, as the above number of benefices also includes the anns, the greatest sum of annual rates paid in any one year, exclusive of marriage-taxes, will be $1011 \times 5.25 = \text{L. } 5307, 15s.$

As several of the not-contributors, at the commencement of the scheme, were below 30 years of age, the number of contributors will not be complete till about the 60th year of the fund; and there being 7 of those alive at Martinmas 1789, the number of contributors may be placed in this form in the succeeding years of the fund, at Martinmas yearly.

Year.	Fund.	Contribut.
1790	47	955
1791	48	955
1792	49	957
1793	50	958
1794	51	959
1796	53	960
1799	56	961
1803	60	962

Although there be several ministers of the church, who also hold offices in the universities; yet this diminution of the number of contributors, such not being liable to double rates, is more than counterbalanced by the number of those who have resigned their benefices or offices, and are still subject to the fund.

From the lists transmitted by presbyteries anno 1742, it appeared, as mentioned above, that out of 897, the greatest number of ministers in life at one

one time, 27 died yearly, &c. According to which proportion, out of 962 ministers and professors, 28.956 would die yearly: of whom 19.3 would leave widows, 5.36 children without a widow, and 4.29 neither widows nor children.

From the trustees report for the 46th year of the fund, ending on the 22d day of November 1789, it appears that there have died of ministers and professors, since the commencement of the scheme,

	1345	quots 29.455,
Which being divided by the } number of years,	45.663	

the number who have died yearly.

Hence the number alive at one
time, = 962.

Divided by the number } who die yearly, =	29.455	quots 32.66,
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the expectation of a minister's life.

From the same report it appears, that 19.3 left widows, 5.8 children without a widow, and 4.4 neither widows nor children, yearly. But seeing, by Dr Halley's table, out of 962 persons of 27 years of age and upwards, 31.79 would die yearly; Dr Webster, in his calculations, supposes that 30 ministers and professors would die yearly; out of whom 20 would leave widows, 6 of them children without a widow, and 4 of them neither widows nor children. He supposes further, that the mean age of widows, at the death of their husbands, was 52, and that the expectation of their life was 20 years.

ART. 3. To find the annual number of those

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contri-

contributors who are liable to the payment of the marriage-tax.

90. As 30 are supposed to die annually out of the whole ministers and professors, of whom 26 leave widows or children; and in regard many wives predecease their husbands, and several of them leave no children; we cannot suppose fewer than 28 to marry annually, who, in consequence of their marriage, except to annuitants, are liable in a sum equal to one year's rate of the class to which they belong; the justness of which inference depends upon the supposition, that all intrants are bachelors, and that there are no second marriages. Therefore, independent of conjecture, it is found in fact, that there are about 26 incumbents who marry yearly in the church and universities; and that there are about 6 married persons admitted yearly, for the first time, to benefices and offices, making in all 32. Besides, of those who are admitted into the church and universities, there are 5 or 6 who are 40 years of age or upwards. Now, supposing that the half of these are either married, or afterwards do marry, and, in consequence of their marriage, are, by the late act, liable in a sum equal to $2\frac{1}{2}$ years rates of the class to which they belong, this will increase the annual number of marriage-taxes to about 35. Hence the sum of marriage-taxes paid annually by contributors will be $35 \times 5.25 = \text{L. } 183, 15 \text{ s.}$ Moreover, 32, the number of yearly marriages, being multiplied by 19.43, the expectation of marriage between two persons of 30 and 25 years of age respectively, will produce 622, the greatest number of married persons in the church and universities at one time; which number being equal

to that of all the widows and widowers in life at one time, and the proportion of these to one another being as 19.4 to 12.6, we shall have $32 : 19.4 :: 622 : 377$, the greatest number of widows in life at one time. See the following Article.

A marriage-tax is payable to the collector, at the first term of Candlemas which shall be one full year after marriage; or, in the case of a married intrant, after his admission to a benefice or office.

ART. 4th, To find the greatest number of annuitants which can come upon the fund. This may be done, 1st, by the annual reports of presbyteries, which, among other articles, contain lists of all the widows alive at Martinmas each year, the medium of which lists for a few years, could we depend upon their accuracy, would shew the maximum of annuitants. Thus, during a space of 20 years from Martinmas, 1749 their mean number was below 380; and the number of widows, annuitants, and others, alive at Martinmas 1789, was 376, including 4, which are added upon account of the great number of families of children of the first order, being 8 more than usual, which came upon the fund at Whitsunday 1781.

2d, By their expectation of life when they commence widows. Let the respective ages of husband and wife be 30 and 25 years; then, to the age of the wife, viz.

	25
add the expectation of marriage, by Dr Halley's	
table	=
	19.5
their sum, increased by .5 on account of second	
marriages, gives 45, the mean age of widowhood;	
therefore 19.56, the expectation of a person's life	
	aged

aged 45, multiplied by 19.3, the number left yearly, will produce 377.5, the maximum of annuitants; whereas 16.66, the expectation of a person's life aged 52, multiplied by 20, the number of widows supposed to be left yearly, produces only 334, which is the number Dr Webster brings out, in the third table of his calculations. So sensible was the Doctor of the defect of this maximum, that, from the year 1781, he superadds, to the number of annuitants by calculation, no fewer than 66, making in all 400 in life at one time.

3d, By the numbers already upon the fund. Thus at Whitsunday 1779, the number of annuitants drawing full was 289.5, including heirs, besides 9 drawing half annuities; at which term there were 181 benefices which had cast no widows upon the fund, but which, in the space of 11 years, produced 34 additional ones, including 4 which are added for a reason mentioned above; at Whitsunday 1790, there were 123 benefices which had produced no widows to the fund. Hence,

181 : 34 :: 123 : 23, the number of additional widows at Whitsunday 1801, making in all 342.5, including heirs, drawing full, and 10 drawing half annuities, when the number of contributors will be complete; at which period there will still be 52 benefices unproductive of widows; hence, $962 - 52 : 342.5 :: 52 : 19.5$ the number of widows still to come upon the fund after Whitsunday 1801. Hence the maximum of annuitants will be 362, including heirs, drawing full, and 10 drawing half annuities; that, is there will be $362 - 5 = 357$ drawing full and 10 drawing half annuities,

annuities, together with 10 not entered upon the fund; in all 377, exclusive of heirs.

It is something remarkable, that although, in assuming the expectation of a widow's life from 52 years and upwards, there is a defect of 2.9 years, yet Dr Webster's calculation, Table III, agrees nearly exact with the fact, till about the year 1781. To account for this I observe,

1st, That the Doctor, in supposing 20 widows to be left yearly, and that these are all alive at the following Martinmas, assumes $.7 + .3 = 1$ more than the fact; which circumstance would increase the number of annuitants by 16, when that number was complete.—2d, That, from the nature of the calculation, he supposes the number of contributors to be complete in the year 1781; whereas there were 15 decliners still alive, and, by proportion, there were 107 original contributors and others still in life, who would produce 37 widows, supposing the maximum to be 334.

3d, That the Doctor's calculation comprehends all the widows alive whether married the second time or not; which circumstance may increase the number about 10 more than the fact, when they arrive at a maximum; now the sum of these, viz. $16 + 37 - 10 = 43$, added to 334, will give 377, the true maximum of annuitants upon the fund; or, subtracted from the true maximum, will leave 334, the number of widows by the Doctor's calculation, Ann. 1781, supposing there had been 962 contributors from the beginning. The induction of these particulars is a further proof that the maximum found in this article is near the truth.

The number of annuitants cannot be complete, not only till after the death of all the decliners, which may happen about the 60th year of the fund, but

but also while any of their successors in office are alive, which may be till about the 120th year of the scheme; during the last 60 years of which period, there are only about 20 additional widows to come upon the fund. Hence the number of annuitants, including heirs, drawing full annuities, at the term of Whitsunday yearly, besides 10 drawing half annuities, may be represented by the following table.

Years.	Fund.	Annu.	Years.	Fund.	Annu.
1790	47	319	1806	63	345
1	48	22	7	64	46
2	49	24	9	66	47
3	50	26	1811	68	48
4	51	28	13	70	49
5	52	330	15	72	350
6	53	32	17	74	51
7	54	34	19	76	52
8	55	36	1821	78	53
9	56	38	24	81	54
1800	57	40	27	84	55
1	58	42	1830	87	56
2	59	43	33	90	57
4	61	344	1863	120	362

A widow's annuity is computed to run from the 26th day of May, or the 22d day of November, which shall be one full half year after the husband's death; and the first year's or half year's annuity is payable on the 26th day of May, which shall be a full year, or a full half year, respectively, after the time from which the annuity is computed to run; and it continues payable on the 26th day of May yearly, during the widow's life, and her remaining unmarried; excepting that, by the late act of parliament, annuitants, having no children under 16 years of age, and marrying contributors,

contributors, are entitled to one half of their annuities, until the capital is completed ; after which they shall enter upon their full annuities ; and as there is nothing due to her heirs for the half year in which an annuitant dies, the heirs only of those who die from Martinmas to Whitsunday are entitled to half years annuities at the following term of Whitsunday.

As a widow is supposed to live at a medium 20 years, did she come upon the fund immediately at her husband's death, her annuity, reckoning interest at $4\frac{1}{2}$ per cent. would be worth 13 years purchase ; but as she hath no claim upon the fund till three quarters of a year at a medium after the above term, and as she loses one quarter of a year at her own death, her annuity is only worth 12 years purchase.

ART 5. Respecting families of children of the first order.

By these are understood those families of children whose fathers at their respective deaths left no widows upon the fund.

89. In calculations respecting the scheme, it is supposed that 6 such families of children are left yearly, who, by act of parliament, are intitled to 10 years purchase of their fathers widows annuity, that is in all, to the sum of L. 1200. And the provision of each family is divided equally among them where there are more children than one ; which provision of children falling due in any one year, computed from the 22d day of November to the 22d day of November in the year following, is payable by the collector, on the 13th day of August thereafter.

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ART. 6. To find the number of families of children of the second order which come yearly upon the fund, that is, children, who being under 16 at the death or marriage of their respective mothers, are intitled to the reversion of what they have not drawn of 10 years annuity.

90. According to lists transmitted by presbyteries, anno 1742, it appeared, as mentioned above, that 3 such families were left annually by widows of ministers who died or married, having children under 16 years of age. Now although the same number die, yet it is probable that fewer marry since, than before the commencement of the scheme, on account of the loss of their annuities; we will suppose that there are 2.5 families of children left annually by widows who died, or married, having a child or children under 16 years of age. But it is evident that no widow who survives her husband 17 years, can leave a family in these circumstances, and by the act of parliament, it is only the children under 16 years of age, of such widows as have not drawn 10 years annuity, or have not survived the 11th year of their widowhood, who are intitled to the reversion; therefore,

$17 : 2.5 :: 11 : 1.6$, the number of families of children of the second order, who are yearly intitled to the reversion of their respective mothers annuity; which reversion is worth from 0 to 10, that is 5 years purchase; Therefore, $1.6 \times 100 = \text{L. } 160$, is the yearly sum due to families of children of the second order.

In this article I differ from Dr Webster, who makes the number of families of children of the second order to be 1.5, because, from the nature
of

of the calculation, the Doctor's number does not comprehend those widows who may happen to marry, having children under 16 years of age; whereas I include these, calling them .5 yearly, seeing they are excluded from the number of widows actually upon the fund. At the same time, I avoid the supposition, that any woman, aged 69 years, can leave a child or children under 16 years of age.

ART. 7. To find the yearly value of the deductions from the provisions of the widows and children of such contributors as happen to die before they have paid a sum equal to three years annuity, corresponding to the annual rate to which they were subjected, without computing interest thereon.

91. Such a deduction will take place, unless a contributor at his death hath paid a sum, including his marriage-tax, amounting to $11\frac{3}{4}$ years rates, or L. 60. Now as contributors, who are admitted into the church or universities at the mean age of 27, cannot be supposed to marry sooner, at a medium, than the third year of their incumbency, or in the 30th year of their age, we may conclude that the proposed deductions will take place only with respect to the families of those contributors who shall happen to die in the third, and in any subsequent year before the 12th of their incumbency, or in the 30th, and in any subsequent year before the 39th of their age. Now, by Dr Halley's-table, out of 16669, the number of persons alive of 27 years of age and upwards, 76 die annually in the 30th to the 38th year of their age, both included; according

to which proportion, 4,386 contributors will die annually in the 30th to the 38th year of their age, both included, out of 962 ministers and professors. But $\frac{1.5}{29.5}$ parts fewer of contributors die yearly, than according to the proportion of the above table, and $\frac{4.4}{29.5}$ parts of those who die, leave neither widows nor children; which $\frac{5.9}{29.5}$ parts of 4,386, subtracted from the same, leave 3,509 the number of contributors dying yearly in the 30th to the 38th year of their age, leaving widows or children; each of whom have paid, including their marriage-tax, from 3 to 11, that is 7 years rates, making in all $24.56 = \text{L. } 128.94$; but they ought to have paid $3,509 \times 60 = \text{L. } 210.54$; therefore, subtracting one from the other, there remain L. 81, 12s. the yearly deductions.

However inconsiderable these deductions* may appear, yet of such consequence were they in the early periods of the fund, that an act of parliament was obtained, anno 1748, authorising the same, in order to prevent a deficiency in the capital of L. 10,000, which was calculated to happen about the year 1771, which defect they have prevented. At the same time it is provided, that every widow shall have at least one half of her annuity for subsistence until the deficiency be made good, and then enter on full payment.

ART. 8. Respecting the annual expences of management.

92. In the first draught of the scheme, which extended only to the ministers of the church, these expences were limited to L. 239. In the two
acts

acts of parliament establishing the fund, wherein the professors in the four universities were also comprehended, the annual expences were fixed at L. 250; namely L. 210 for the salaries of the general collector and clerk, and L. 40 for the annual expences of the trustees. By the late act of parliament anno 1778, the salaries of the collector and clerk are the same as formerly, but the incidental expences of the trustees are augmented to L. 70, with a proviso that they do not in any one year exceed that sum. Which expences of the trustees, in imitation of Dr Webster, are not brought as a charge upon the fund, from the supposition that the interest received by the collector in any one year, doth so far at least overbalance the interest of the capital for that year, computed at 4 per cent.

ART. 9. Respecting the capital of the fund.

93. Had there been no capital, the rates being adjusted to the annual demands upon the fund, it is evident that during the first 14 years of the scheme the annual rates would have been too low, and that afterwards, till the number of annuitants arrived at a maximum, they would have risen gradually to L. 8.5 at a medium, which is out of all proportion to the value of an annuity of L. 20. By the act of parliament establishing the fund, anno 1744, when the maximum of annuitants was supposed not to exceed 321, the capital was limited to L. 63,860, including the loans of L. 30 lent to contributors; the interest of which, together with the annual rates and taxes, was thought sufficient to discharge all demands upon the fund. By the act of parliament, anno 1748, when it was found that the
number

number of annuitants would exceed 364, the capital was extended to L. 80,000, including the loans of L. 30, granted to contributors at 4 per cent. And by the late act, anno 1778, when it was supposed that the number of widows might rise to 400, the capital is allowed to be raised to the sum of L. 100,000, to be lent out upon heritable security, hitherto at $4\frac{1}{2}$ per cent.; and the loans of L. 30 to contributors are prohibited, which of consequence will all revert to the capital about the 95th year of the fund. And in order to secure the rise of the capital to L. 100,000, the trustees are directed to take care that the sum of L. 200, together with the savings and surpluses of interest, annual rates, &c. that these, after all demands upon the fund are discharged, be applied annually for that purpose, by lending out the same, with the advice and consent of the learned judges who preside in our courts of law, until the capital is made up. When it is completed, there will be an overplus, reckoning interest at $4\frac{1}{2}$ per cent. after paying all demands upon the fund, of about L. 1400 yearly; of which L. 150 must be reserved as a provision for those 18 widows who are still to come upon the fund: as for the remainder, the church must consult and determine in what manner it ought to be disposed of, when the capital is raised to L. 97,000.

94. ART. 10. To trace the progress of the capital from L. 87751.522, in the 46th year of the fund, till it arise to L. 100,000. Here,

Art.

Art.

2. The number of contributors, being
from 955 to 962.
3. — of persons liable to the marriage-tax, 35.
2. — of vacancies, 49.
4. — of annuitants, including heirs, drawing full, besides 10 drawing half annuities, being from 319 to 362.
5. — of families of children of the first order, being 6, = L. 1200.
6. — of families of children of the second order, being 1.6, = L. 160.
7. Deductions from the provisions of widows and children, = L. 81.6.
8. The salaries of the collector and clerk, = L. 210.
1. The medium of annual rates, L. 5.25; and of annuities, L. 20; and the rate of interest upon the capital 4 per cent. The progress of the capital may be represented by the following table.

I.	II.	III.	IV.	V.	VI.	VII.
1789	46	Rates.	their prod.	Ann.	their exp.	87,751.522
1790	47	1039	5454.75	319	7968.4	88,747.932
1791	48	1039	5454.75	322	8028.4	89,724.2
1792	49	1040	5460.	324	8068.4	90,704.768
1793	50	1041	5465.25	326	8108.4	91,689.809
1794	51	1042	5470.5	328	8148.4	92,679.5
1795	52	1043	5475.75	330	8188.4	93,674.03
1796	53	1043	5475.75	332	8228.4	94,668.34
1797	54	1044	5481.	334	8268.4	95,667.67
1798	55	1044	5481.	336	8308.4	96,666.97
1799	56	1044	5481.	338	8348.4	97,666.25
1800	57	1045	5486.25	340	8388.4	98,670.75
1801	58	1045	5486.25	342	8428.4	99,675.43
1802	59	1045	5486.25	343	8448.4	100,700.3
1803	60	1045	5486.25	343	8448.4	101,762.16
1804	61	1046	5491.5	344	8468.4	102,855.75

In

In this Table, Column VII. shews the amount of the free stock at Whitfunday for each corresponding year of the fund in Col. II. the interest of which, computed at 4 per cent. together with the sum of the rates and taxes of the following year, Col. IV. is added to said stock, and their sum, diminished by the charge upon the fund, Col. VI. is the free stock of the following year. Col. III. shews the number of rates and taxes due at Candlesmas, for each year of the fund; and Col. V. the number of annuitants, including heirs, drawing full, besides 10 drawing half annuities at the term of Whitfunday yearly.

Ex. To stock at Whitfunday 1801, = L. 99675.43
 Add interest to Whitfunday 1802, = 3987.02
 And 1045 rates and taxes, = 5486.25

Total,	=	L. 109148.7
Deduct annuities diminished by deductions, Art. 7.	=	L. 6878.4
Childrens provisions and salaries,	=	1570.0
		8448.4

Stock at Whitfunday 1802, = L. 100700.3

Thus, about the 61st year of the fund, the capital will be made up, the number of contributors will be complete, and there will be 18 widows still to come upon the fund in the course of 60 years.

ART. II. Respecting alterations upon the scheme.

97. 1st, It is required, to find what capital would

would be necessary to support the fund upon its present footing, when the annuitants are arrived at a maximum, reckoning interest at 4 per cent.

From the widows annu-

ties, = L. 7340

Childrens provision and sa-

laries, = 1570

Their sum, = 8910

Deduce rates, taxes, and

deductions, = 5573.1

L.

Divide by .04) 3336.9 (83422.5, the Capital.

2d, It is required, to find what diminution might be made upon the annual rates, supposing the annuities to remain the same as at present, when the capital is raised to L. 100,000, reckoning interest at $4\frac{1}{2}$ per cent.

To 357 annuitants drawing

full, = L. 7140

Add 20, including heirs,

drawing half annuities, = 200

Childrens provision, = 1360

And salaries, = 280

Their sum, = 8980

Deduce interest and deduc-

tions, = 4581.6

L.

Divide by rates and taxes,) 4398.4 (4.2, Annual Rate.

Here, $1046 \times 5.25 - 4398.4 = \text{L. } 1093.1$, is the annual surplus.

3d, It is required, to find what augmentation might be made upon the annuities, the annual rates remaining the same as at present, when the capital is raised to L. 100,000, and reckoning interest at $4\frac{1}{2}$ per cent.

From interest, rates, and de-

ductions,	=	10073.1	
Deduce salaries,	=	280.0	
		—————	L.

Divide by annuitants and		) 9793.1	(22.51, the
children,	= 435		

ART. 12. Containing a general view of the scheme.

96. 1st, It is peculiar to this scheme, that the annual rate varies, not with the age of a contributor, but only in proportion to the annuity; that every intrant who is married, or afterwards marries, is liable to a marriage-tax; that its construction is built upon the principles of the expectation of a minister's life, which, being above 34 years, including the vacancy, with a high rate of interest, depresses the rates as much as possible; that the widow's annuity begins to run, from a full twelvemonth after the last payment of the husband; that widows who happen to marry are deprived of the whole, or of part of their annuities; and it is owing to this scheme that the distinction of families of children of the second order was ever heard of.

2d, As to the disadvantages attending this scheme, if there be any, they are such as are in common with all laudable undertakings of this kind.

kind. Thus the leading principle of such a scheme not being justice and equity, but charity, mercy, and compassion for the indigent; the more you pay, the less is received, and the less you pay, the greater is the value of the annuity. As all intrants are obliged to contribute for the support of the fund, this circumstance, though an advantage to the scheme, is often a hardship upon the individual, who is thus obliged to provide for a widow, when possibly he may have no thoughts of taking a wife. As heirs in general distinct from widows and children have no interest in the fund, the contributions of $\frac{4}{30}$ parts of those who die yearly are lost to them and their heirs for ever, if what is given for the support of the widow and fatherless can be called a loss. The depriving a widow of her annuity in consequence of a second marriage, may be considered as a disadvantage attending the scheme, not from a principle of justice or charity, seeing every woman who marries is supposed to be supported by her husband, but in a political view, as the increase of society is thereby so far discouraged. And the capital of the fund, like all other sums of money laid out at interest, is liable to a diminution in its value, from the possible fall of the interest of money. But I insist, with much more pleasure,

3d, Upon the advantages attending the scheme; as hereby, in common with other institutions of the same kind, so many widows and orphans receive the means either of subsistence or of education, which must prove a benefit to contributors themselves, who, in consequence of the scheme, are enabled to marry with greater advantage than

they could have done had it not taken place. The annual rates are exceeding low, on account of the advantages which the scheme itself enjoys; as, 1st, All entrants being subject to the scheme, they enter early, and of consequence have a chance of continuing long contributors to the fund. 2d, Bachelors and widowers being liable to the payment of the annual rates, the heirs of a seventh part of those who die yearly derive no advantage from the fund. 3d, Anns and vacant stipends, where there is no waste in the life of the incumbent, are subjected to the payment of half years rates. 4th, There are a number of taxes and deductions, see Art. 1. which are not supposed in the calculation of the rates. 5th, Should a contributor fail in the payment of his rates, which are privileged debts, there is a visible fund, namely, his stipend or salary, out of which the scheme may be reimbursed for all losses occasioned by him; and should this fail, the collector hath 26 chances out of 30, of recovering at 5 per cent. simple interest, all arrears due by a contributor, out of his widow's or childrens provision. 6th, The rise of the capital to a certain amount, which is lent out upon heritable security, and its recovery, should it sustain any material loss, are both secured by act of parliament. 7th, The number of contributors is fixed and determined, being limited to the number of benefices in the church and offices in the four universities; so that the trustees are enabled to foresee what demands are likely to be made upon the fund from year to year. 8th, The scheme itself hath undergone several alterations and improvements, and may still undergo more by favour of the British legislature. 9th, The whole concerns of the scheme are under the
the

the direction and management of trustees, most respectable for their knowledge, experience, and perseverance in the affairs of the fund; namely, the Reverend Ministers of the presbytery, and learned Professors in the university of Edinburgh, together with the ministers in the presbytery seats, or who are admitted to offices in the universities, if they chose to accept, and delegates from presbyteries and universities, who, did they all convene, would form a general court of directors, as respectable for their numbers and importance as the Venerable Assembly itself.

Therefore, I cannot conclude without expressing my respect and gratitude to the Reverend and learned Trustees, who are appointed to manage and direct the operations of the fund, and thanking them for their fidelity and unwearied application in conducting the concerns of a scheme, whereby the widow's heart is made to sing for joy.

SUPPLEMENT to CHAPTER IX.

A

DISSERTATION

RESPECTING

The DISPOSAL of the SURPLUS in the Fund of the WIDOWS SCHEME, when the Capital is raised to L. 100,000; addressed to the Trustees for managing said Fund.

MR PRESES,

99. THE question now under consideration, respecting the disposal of the surplus in the fund of the widows scheme, is of a similar nature with, but of a different tendency from that determined by the church, anno 1748, and of consequence ought to have a similar though opposite determination. In the year 1748, the Reverend Dr Alexander Webster, the principal contriver and promoter of the widows scheme in the Church of Scotland, discovered by calculation, that there would be a deficiency in the capital of the fund, of L. 10,000 about the year 1771. What was then to be done to prevent this deficiency? Must the rates be raised? No; Must the annuities be diminished?

nished? No; Touch not, would he say, either of these pillars of the scheme, left by sapping and undermining the foundation, you make the whole fabric tumble about your ears. What then? Why, the Doctor did the only thing which he ought to have done; he did an act of justice, by obliging every contributor to pay at least three years purchase of the annuity corresponding to the rate to which he was subjected, before his widow or children could be entitled to their respective provisions.

In the present case, when the capital is raised to L. 100,000, the annual surplus may be about L. 1473, reckoning interest at $4\frac{1}{2}$ per cent.* What must be done to prevent the farther increase of the capital? Shall we diminish the rates in general? or shall we increase the annuities? Either, or both of these would be desirable, did we not thereby perpetuate a very great hardship upon aged contributors, by continuing the payment of the rates till the close of life; when we have it in our power to redress it.—But,

To find how this matter may be determined to the satisfaction both of contributors and annuitants, by giving each party a share of the surplus, nearly in proportion to their numbers. There may be 435 annuitants upon the fund, when they arrive

* When the capital is complete, an. 1802, there may be 338 annuitants drawing full, and 20, including heirs, half annuities; Hence to rates, taxes, and deductions, = L. 5573
Add the interest of the capital at $4\frac{1}{2}$ per cent. = 4500

	Their sum	=	L. 10073
To the sum of annuities paid to widows,	=	L. 6960	} 8600
Add chil. prov. and expence of management	=	1640	
Their difference is the annual surplus	=	L. 1473	at

at a maximum, including both orders of children, which are equal in expence to 68 widows; and the number of contributors alive at one time is 962.

Now it is obvious that annuitants of all descriptions, whether old or young, are equally the objects of charity, and of consequence, ought to be benefited only by the plan of augmentation; suppose by 5 per cent, or by 1 shilling upon the pound of their annuities; this will exhaust L. 435 of the above surplus. Again,

With respect to contributors, in the relation they stand in to the scheme, by the payment of the rates, &c. take the following table.

<i>Years of minr.</i>	<i>Contributors.</i>	<i>Amount at 4 per cent.</i>	<i>Yrs. pur.</i>
From 0 to 12	3.6 fail who pay	L. 67.5 for an annuity worth	15.0
12 to 28	74	149	13.5
28 to 65	18.45.	524	10.0
0 to 33	15.0	132	14.0
33 to 65	15.0	605	10.0

By this table, out of those who are alive, during the first 12 years of their ministry, 3.6 fail yearly, who, allowing 5s. for expence of management, have paid at a medium, including deductions, the sum of L. 67.5, for a L. 20 annuity worth 15 years purchase. It would therefore be highly improper to grant a diminution of the rate during this stage of life, not only on account of the small value paid for the annuity, but also because the more the rate was diminished, the greater would be the deduction out of the widows provision. In like manner, in the case of those who are alive from the 12th to the end of the 28th year of their ministry *, when the full value of the annuity is

* To find the time in which a rate of L. 5 per annum, allowing 5s. for expence of management, will discharge the augmented annuity of L. 21, worth 12 years-purchase, at 4 per cent. $\frac{S}{a} = \frac{252}{5} = 50.4 \approx N$ corresponding to 28.13 years.

discharged

discharged, it would also be unnecessary to diminish the rate during this period, because contributors have not paid the full value of the annuity. But in the case of those who are alive from the 28th year of their ministry to the close of life, it is manifest that, during this interval, there ought to be a total exemption from the payment of the rate, as far as the above surplus of $1473 - 435 =$ L. 1038 will admit; seeing these contributors have paid, at a medium, the sum of L. 324 more than the value of the annuity. Therefore,

I call upon you, Mr Preses, I call upon the church in general, seeing you have it now in your power, to do an act of justice to our aged brethren, who have borne and still will bear the burden in the heat of the day, by suspending the payment of the rates after a certain age, or rather, by exempting aged contributors after a certain number of annual payments, including the marriage-tax, be that number what it will; which number, could I depend upon the only table of observations which approaches near to the better sort of lives in Scotland, might be 30*, including the marriage-tax; upon the supposition that each contributor thus exempted was obliged to pay, in name of vassalage, 5 per cent. or 1 s. upon the pound,

* It is required, to find, if L. 938 of annual surplus will admit of a suspension of the rates beyond 30 annual payments, including the marriage-tax. Supposing 30 the mean age of admission, there are alive, by the Fife table of observations, at the age of 30 years, $358 - 2 = 356$, and at 60 years of age, $211 - 4.5 = 206.5$, whose expectation of life is 12.98; and there are admitted into the church and universities 29.455 contributors yearly. Hence, $356 : 206.5 :: 29.455 : 17.08$, and $12.98 \times 17.08 = 222$, alive at 60 and upwards. Hence, $222 \times 4.25 =$ L. 943.5, the sum remitted to these contributors.

of the annuity corresponding to the rate to which he was subjected, and that L. 100 of surplus was reserved, in order to co-operate with the savings upon the rates and annuities, till the ex-emptees and annuitants arrived at a maximum, in forming an additional fund of L. 9000, to discharge the annuities of those 18 widows who are still to be provided for, as appears from the following

T A B L E.

Age.	Rate.	Living.	Expe ^{ct} .	Total alive.
30	0	29.455	×	32.66 = 962.
42	12	25.85	×	24.29 = 628.
58	28	18.45	×	14.00 = 258.
59	29	17.79	×	13.50 = 240.
60	30	17.08	×	12.98 = 222.
61	31	16.34	×	12.53 = 205.
62	32	15.60	×	12.10 = 189.
63	33	14.85	×	11.66 = 173.

If the capital be allowed to rise to L. 109,000, as suggested above, by the accumulation of L. 150 per annum, without interest, during the space of 60 years, when the annuitants will arrive at a maximum: in this case, the above surplus of L. 1038, appropriated to contributors, with a tax of 1 s. per pound of the annuity, will reduce the years of exemption to 28, when a contributor hath paid the full value of the augmented annuity, and when the fourth part of those alive at
one

one time in the church will be exempted annually. Moreover,

If the annual surplus shall happen to rise considerably above L. 1000, both plans of augmentation and exemption may be used to advantage; but should it not exceed or fall below that sum, the plan of exemption ought alone to be employed.—Now, that such a plan ought to be adopted, in preference to all others, will appear,

1st, From its being founded upon a principle of justice. In such a Widows Scheme as this, we acknowledge that charity, mercy, or compassion for the indigent, are the leading principles: therefore, as long as it is necessary for the existence of the scheme, rates ought to be paid till the close of life; but when this hardship, as in the present case, can be removed, and justice mixed with mercy, it ought to be done. With what confidence can we presume to diminish the rates in general, or augment the annuities, whilst we leave this burden upon aged contributors unremoved? A grievance which, remaining unredressed, must blast the reputation of every other plan, at least in the opinion of every discerning person. Besides, every one who is exempted after 30 annual payments of the rate, will have contributed a sum, viz. L. 295, whose interest, at 4 per cent. is with in a trifle of L. 12 per annum; whilst the other contributors, from 30 years and under, will have only paid, each of them, at a medium, a sum, viz. L. 124, whose interest, together with the annual rate, does not exceed L. 10.2 per annum: so that the former benefit the scheme as much after an exemption, as the latter do with the payment of the rate.

2d, From its possessing the beneficent properties of all other plans. It prevents the accumulation of the capital beyond a limited sum; it augments the annuities with L. 435, and diminishes the rates to the extent of L. 1038 per annum, and that too most effectually where there is most occasion for a diminution; and as a penny saved is so much gained, it may be said to augment the annuity, in the case of exemptees, by the value of the rate during the life of the contributor. Moreover, was it thought adviseable to credit the trustees with the small discretionary power of augmenting or diminishing the years of exemption, according as it should be found in fact, that the number of annuitants or of exemptees exceeded or fell short of the calculation, or that the interest of money was below $4\frac{1}{2}$ per cent.; this plan might keep the affairs of the scheme upon such an even footing, as to prevent all future applications to parliament for redress of grievances, which cannot fail of being both expensive and troublesome; whilst the losses sustained by bad debts or otherwise are effectually guarded against by the law now in force.

3d, From the simplicity of its construction and operation. Let any one attempt to diminish the rates or augment the annuities, he will find more difficulties than possibly he expected, and will be in danger of unhinging the whole frame of the scheme. But by this plan, the annuities are paid in guineas, in place of as many pounds Sterling, and the rates remain entire as from the commencement of the fund, which still enjoys the full benefit of deductions and of half-years rates out of

of anns and vacant stipends; so that the affairs of the scheme, in general, proceed in their usual course, which cannot fail of being a great advantage, in point of ease, both to the trustees and general collectors. The only calculation necessary will be to determine the years of exemption; for which purpose had we a table of observations adapted to the lives of ministers, this might be done with accuracy; if not it may be discovered from fact, with this caution, that the number of exemptees will not arrive at a maximum till about the 85th year of the scheme. But let none object to this plan,

1st, That I am in danger of losing the concurrence of the younger members of the church, who form the greater part of the society. No; I will cast myself upon their generosity: Dr Webster did so at the commencement of the scheme, and was not disappointed. Had it not been for their disinterested conduct, the scheme would never have taken place; nor can I suppose that, in the course of half a century, this public spirit should be degenerated in the younger members of the society. But I ask less than the Doctor did; I only ask an act of justice towards their aged brethren, whilst the youngest in the church have at least a chance of reaping the same benefit with the oldest, and many may there be who fulfil the 30th year of their ministry, before which they have no cause to complain, not having discharged the full value of the annuity. Nor,

2^d, That this plan comes from an obscure corner of the church, and from an inconsiderable hand.